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BY

C. H. BRADBURY

A.M.Inst.C.E., A.M.I.Mech.E., A.M.I.A.E.

With 57 Illustrations (One Folding Plate)

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LIST OF SYMBOLS USED.

a	Amplitude.
	A constant.
b	Disturbing force a_1/I.
c	A constant.
d	Diameter.
e	Hyperbolic constant.
f	Natural frequency.
g	32·2.
h	Half number of ordinates (Harmonic Analysis).
i	$\sqrt{-1}$.
j	Frequency (squared).
l	Length.
m	Mass.
n	Number.
p	$\mu/2I$.
q	Forcing phase velocity.
r	Radius.
	Radius of gyration.
s	A constant.
t	Time.
v	Velocity.
w	Width.
δw	Work done per vibration.
A	Area.
	A constant.
B	A constant.
C	A constant.
D	Diameter.
E	Kinetic energy.
	E.M.F. of generator.
G	Torsional modulus.
H	Harmonic component.
I	Moment of inertia.
I_a	Current.
K	$2p/\omega$.
L	Length.
M	Dynamic magnifier.
N	R.P.M.
P	Maximum amplitude of forced vibration.
R	Shaft stiffness.
S	Torsional stress.
T	Torque.
U	Frequency constant.
V	Frequency constant.
W	Weight.
X	$\omega\sqrt{I_1/R_1}$.
Z	Hysteresis constant.
Z_T	Section modulus in torsion.
α	Angle.
	Amplitude.
β	A constant.
γ	Hysteresis exponent.
δ	Equilibrium amplitude.
ε	Angle of lag.
η	$W\omega^2 r/g$.
θ	Angle.
λ	l_n/l_1.
μ	Damping factor.
π	3·1416.
ρ	I_n/I_1.
	Gear ratio.
τ	Periodic time.
φ	Phase angle.
ω	Natural phase velocity.
Φ	Function.
Σ	Summation of.

TORSIONAL VIBRATION IN DIESEL ENGINES

INTRODUCTION

Torsional vibration in diesel engines is due primarily to the fact that the periodically varying cylinder torque acts upon an elastic shaft system which itself has one or more inherent natural frequencies, depending upon the proportions of the shaft and the number and distribution of the attached masses. Each frequency is distinguished by the number of nodes formed in the shaft during vibration, the lowest frequency being that associated with one node, the lowest but one with two nodes and so on.

When the shaft is rotating at such a speed that the power strokes of the engine synchronise with one of the natural frequencies of the shaft system, it is said to be in a critical speed. When the speed is such that the natural frequency is exactly twice the number of revolutions per second, the shaft is said to be in a critical speed of the second order. Thus, in a four cycle engine, it is possible to have critical speeds of every integral order and half order and in a two cycle engine critical speeds of every integral order, the order being determined by the number of vibrations occurring per revolution of the engine.

Critical speeds are not all of equal importance. In general, the most important are those the torque components of which act cumulatively, so as to twist the shaft in its normal mode of vibration. These are known as " major " criticals, all others being termed " minor " criticals. The major critical speeds most frequently encountered are those the order of which is a direct multiple of the number of power strokes per revolution. In the majority of cases, minor critical speeds are not important, though in some instances they may set up stresses sufficiently high to fracture a crankshaft and for this reason they must always be taken into consideration. The actual stress set up by a critical speed is dependent on a number of factors which will be considered later.

THE TORSIONAL PENDULUM

Free Harmonic Vibrations.—The torsional pendulum, which is the simplest example of a system capable of torsional vibration, consists essentially of a disc supported by a shaft as shown in Fig. 1.

If the disc is given a small initial twist and then suddenly released, torsional vibrations will ensue. These vibrations, which are maintained by the elastic restoring forces in the shaft, are known as the " free " or " natural " vibrations of the system and an expression for their characteristics can be found by setting down the differential equation of motion of the system and obtaining its solution.

Let I = the polar moment of inertia of the disc. For a uniform disc of radius of gyration r ft. and weight W lbs.

$$I = \frac{Wr^2}{g} \text{ lbs. ft.}^2 \text{ sec.}^2$$

Let R = the stiffness of the shaft. For a uniform shaft of length l ft. and diameter d ft., the material of which has a torsional elastic modulus G lbs. per sq. ft.

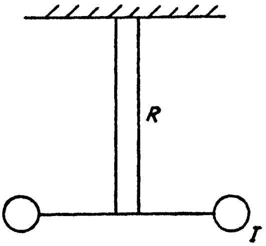

Fig. 1.—Torsional Pendulum.

$$R = \frac{G\pi d^4}{32l} \text{ ft. lbs. per radian.}$$

Let α = the angle of twist at any instant. Then the torque moment in the shaft at any instant will be $R\alpha$. Neglecting the mass of the shaft,* this torque moment will be equal in magnitude but opposite in sign to the moment of inertia of the disc times its angular acceleration, whence

---

* In nearly all cases, it is possible to neglect the mass of the shaft when dealing with torsional vibration in diesel engines, as the effect on the frequency is very small. If it is desired to take the shaft into account, one third of the moment of inertia of the shaft should be added to that of the mass.

$$I\frac{d^2\alpha}{dt^2} + R\alpha = 0$$

or

$$I\ddot{\alpha} + R\alpha = 0 \quad . \quad . \quad . \quad (1)$$

Dividing through by I and letting $\frac{R}{I} = \omega^2$

$$\ddot{\alpha} + \omega^2\alpha = 0 \quad . \quad . \quad . \quad (2)$$

The solution of this equation is

$$\alpha = C_1e^{i\omega t} + C_2e^{-i\omega t} \text{ where } i = \sqrt{-1}$$
$$= A \cos \omega t + B \sin \omega t \quad . \quad . \quad . \quad (3)$$

where A and B, which equal $(C_1 + C_2)$ and $i(C_1 - C_2)$ respectively, are arbitrary constants.

This solution shows that the vibration of the disc is Simple Harmonic Motion.

By writing $A = a \sin \varphi$ and $B = a \cos \varphi$ we have

$$\alpha = a \sin (\omega t + \varphi) \quad . \quad . \quad (4)$$

This is the general equation for Simple Harmonic Motion, in which ω is the phase velocity, usually in radians per second, a is the amplitude of the oscillation and φ is the phase angle when $t = 0$.

Now since $\omega^2 = \frac{R}{I}$, the motion of the disc is given by

$$\alpha = a \sin \left(\sqrt{\frac{R}{I}}t + \varphi\right) \quad . \quad . \quad (5)$$

and since ω is the phase velocity, the time of one complete vibration is

$$\tau = \frac{2\pi}{\omega} \text{ seconds}$$

and the frequency of the vibrations

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi}$$

$$= \frac{1}{2\pi}\sqrt{\frac{R}{I}} \text{ vibrations per second} \quad . \quad (6)$$

Free Vibrations with Damping.—In the case already considered, the effect of damping has been neglected, and it was assumed on page 2 that once the torsional pendulum has been disturbed, the vibrations will continue indefinitely with constant amplitude. In actual practice damping, in the form of hysteresis in the material, air resistance, etc., is always present, and the natural vibrations gradually decrease in amplitude until the pendulum finally comes to rest. The actual relationship between this damping force and the other factors involved is at present an open point, but in dealing with the torsional pendulum, it is usually assumed for mathematical convenience that the damping is proportional to the velocity of the vibrations.

Therefore if μ is the coefficient of friction in foot-lbs. per radian per second, the damping force is $\mu\frac{d\alpha}{dt}$.

The differential equation for damped free vibration now becomes

$$I\frac{d^2\alpha}{dt^2} + \mu\frac{d\alpha}{dt} + R\alpha = 0$$

or

$$I\ddot{\alpha} + \mu\dot{\alpha} + R\alpha = 0 \quad . \quad . \quad (7)$$

Dividing through by I and letting

$$\frac{\mu}{I} = 2p, \quad \frac{R}{I} = \omega^2$$

we have

$$\ddot{\alpha} + 2p\dot{\alpha} + \omega^2\alpha = 0 \quad . \quad . \quad . \quad (8)$$

The general solution of this equation is

$$\alpha = e^{-pt}\left(C_1 e^{i\sqrt{\omega^2 - p^2}t} + C_2 e^{-i\sqrt{\omega^2 - p^2}t}\right)$$

provided that p^2 is less than ω^2. If p^2 is not less than ω^2 the motion is dead beat and non-vibratory.

Therefore

$$\alpha = e^{-pt}(A \cos \sqrt{\omega^2 - p^2}t + B \sin \sqrt{\omega^2 - p^2}t) \qquad (9)$$

where A and B, which equal $(C_1 + C_2)$ and $i(C_1 - C_2)$ respectively, are arbitrary constants.

This represents simple harmonic motion whose amplitude diminishes asymptotically to zero.

Equation (9) can be represented as simple sinusoidal motion.

Let $A = a \sin \varphi$ and $B = a \cos \varphi$ where a and φ are constants as in equation (4).

Then $\alpha = e^{-pt}\{a \sin (\sqrt{\omega^2 - p^2}t + \varphi)\}$ . . (10)

As in the case of Equation (5) the natural frequency is given by

$$f = \frac{1}{2\pi}\sqrt{\omega^2 - p^2} \quad . \quad . \quad . \quad (11)$$

If p is small compared with ω, p^2 can be neglected, and the frequency then becomes the same as that for undamped simple harmonic motion. It may thus be assumed that the natural frequency is unaffected by the presence of a small damping coefficient.

Substituting $\frac{R}{I}$ for ω^2 and $\frac{\mu}{2I}$ for p we have

$$f = \frac{1}{2\pi}\sqrt{\frac{R}{I} - \left(\frac{\mu}{2I}\right)^2} \text{ vibrations per second} \qquad (12)$$

The "rate of diminution of amplitude" of the damped torsional pendulum can be very conveniently expressed in terms of the "logarithmic decrement." Referring to Fig. 2, which shows the amplitude plotted against ωt, the time between A and B is

$$t = \frac{2\pi}{\omega} \text{ seconds}$$

In this time, therefore, the amplitude drops from

$$e^{-pt} \text{ to } e^{-p\left(t+\frac{2\pi}{\omega}\right)}$$

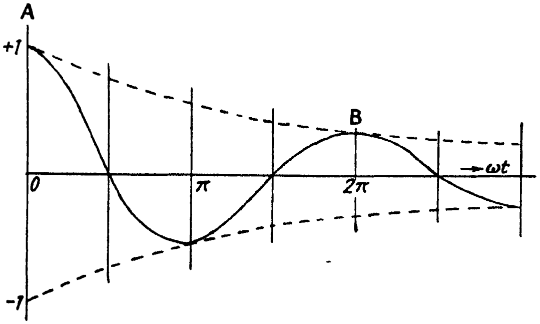

Fig. 2.—Logarithmic Decrement.

The second amplitude is the first multiplied by $e^{-\frac{2\pi p}{\omega}}$ and since this factor is the same for any two consecutive maxima, the amplitudes decrease in geometric series.

But numbers in geometric series have their logarithms in arithmetic series. Thus the difference between the logarithms of two consecutive maxima, *i.e.* $\frac{2\pi p}{\omega}$, is constant. This is known as the "logarithmic decrement."

Substituting $\frac{\mu}{I}$ for $2p$ we have

$$\text{Logarithmic Decrement} = \frac{\pi\mu}{\omega I} \quad . \quad . \quad (18)$$

Forced Vibrations with Damping.—Consider now the case when the damped simple harmonic motion of Equation (7) is acted upon by a periodic disturbing force of the form $a_1 \sin qt$.

The differential equation of motion now becomes

$$I\frac{d^2\alpha}{dt^2} + \mu\frac{d\alpha}{dt} + R\alpha = a_1 \sin qt$$

or

$$I\ddot{\alpha} + \mu\dot{\alpha} + R\alpha = a_1 \sin qt \quad . \quad . \quad (14)$$

Dividing through by I and letting

$$\frac{\mu}{I} = 2p, \ \frac{R}{I} = \omega^2 \text{ and } \frac{a_1}{I} = b$$

we have

$$\ddot{\alpha} + 2p\dot{\alpha} + \omega^2\alpha = b \sin qt \quad . \quad . \quad (15)$$

The complete solution of this equation is

$$\alpha = e^{-pt}\{a \sin (\sqrt{\omega^2 - p^2}t + \varphi)\} + P \sin (qt - \varepsilon) \quad . \quad (16)$$

where

$$P = \frac{b}{\sqrt{(\omega^2 - q^2)^2 + 4p^2q^2}} \quad . \quad . \quad (17)$$

and

$$\tan \varepsilon = \frac{2pq}{(\omega^2 - q^2)} \quad . \quad . \quad . \quad (18)$$

The first term on the right-hand side of Equation (16) represents the natural vibrations, previously considered, which are soon damped out. The second term represents the forced vibrations.

When the natural vibrations have been damped out

$$\alpha = P \sin (qt - \varepsilon)$$

This is simple harmonic motion of the same frequency as the disturbing force, and lagging an angle ε behind it. The maximum amplitude is P.

Now it will be remembered that $b = \frac{a_1}{I}$ and $\omega^2 = \frac{R}{I}$ so that $\frac{b}{\omega^2} = \frac{a_1}{R}$.

Now $\frac{a_1}{R}$ is the maximum value of the disturbing force divided by the shaft stiffness. This ratio thus represents the amplitude which would be produced by the steady application of the force a_1.

Therefore the maximum " static " amplitude is

$$\delta = \frac{b}{\omega^2} \quad . \quad . \quad . \quad (19)$$

Dividing the top and bottom of Equation (17) by ω^2, the maximum " dynamic " amplitude becomes

$$P = \frac{\delta}{\sqrt{\left(1 - \frac{q^2}{\omega^2}\right)^2 + \frac{4p^2q^2}{\omega^4}}} = \frac{\delta}{\sqrt{\left(1 - \frac{f_1^2}{f^2}\right)^2 + \frac{K^2f_1^2}{f^2}}} \quad (20)$$

where $f = \omega/2\pi$ = the natural frequency.
$f_1 = q/2\pi$ = the forcing frequency.
$K = 2p/\omega$ = a constant depending on the damping forces.

The fraction

$$M = \frac{1}{\sqrt{\left(1 - \frac{f_1^2}{f^2}\right)^2 + \frac{K^2f_1^2}{f^2}}} \quad . \quad . \quad (21)$$

therefore represents the degree of magnification of the static amplitude due to the relationship between the three factors f, f_1 and K. It is commonly known as the " dynamic magnifier."

If f_1 is very small compared with f, $M = 1$ and the " dynamic " amplitude equals the " static " amplitude, *i.e.* the amplitude may be calculated on the basis that the disturbing force is steadily applied.

If $f_1 = f$, $M = 1/K$. Thus if the damping is small, the dynamic amplitude is very large and under these conditions a small disturbing force may produce an amplitude of sufficient magnitude to fracture the shaft. This is the important condition of resonance.

If f_1 is very large compared with f, M tends to equal zero, and the dynamic amplitude thus actually becomes much smaller than the static amplitude.

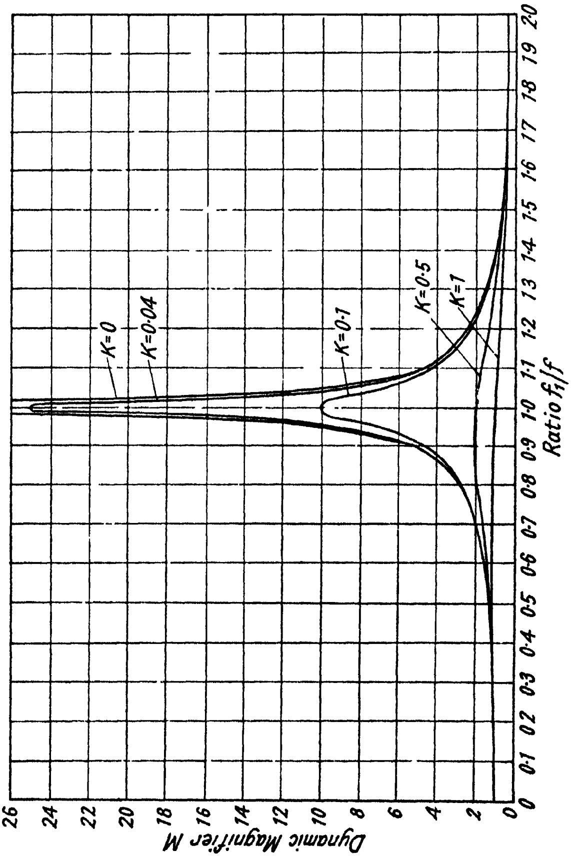

Fig. 3.—Relationship between Dynamic Magnifier and Ratio f_1/f.

These three conditions are most clearly shown in the form of a diagram. Fig. 8 shows the dynamic magnifier M (Equation 21) plotted against the ratio f_1/f for various values of K. What may be termed the " danger zone " is clearly seen to lie approximately between the ratios $f_1/f = 0{\cdot}8$ and $1{\cdot}20$.

As the ratio f_1/f varies, the angle of lag ε also varies. Referring to Equation (18), if q is small compared with ω, the angle of lag is small. If $q = \omega$, the condition for

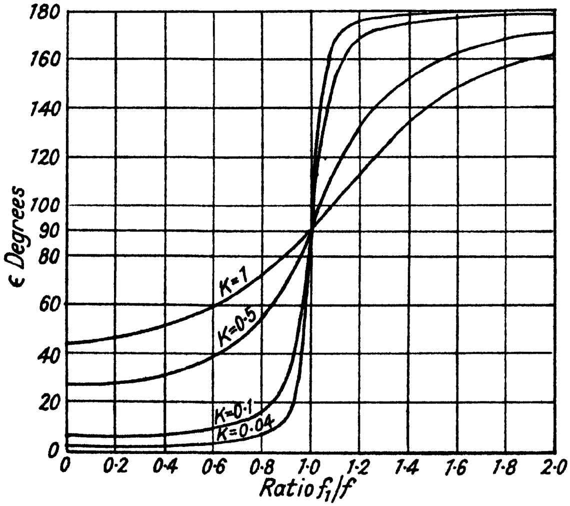

Fig. 4.—Relationship between Angle of Lag ε and Ratio f_1/f.

resonance, $\varepsilon = \pi/2$, *i.e.* the phase of the vibrations is 90° behind the disturbing force. If q is large compared with ω, tan ε is negative, *i.e.* the phase is opposite to that of the disturbing force.

Fig. 4 shows these conditions diagrammatically and it will be seen that for low values of the damping forces there is a very rapid change in phase as resonance is approached.

THE NATURAL FREQUENCIES OF MULTI-MASS SYSTEMS

The Frequency Equation.—The torsional pendulum considered on page 2 forms a basis for all diesel engine natural frequency calculations, but only in extreme cases such as a single cylinder engine with a heavy fly-wheel or a radial aero engine with a heavy airscrew is it directly applicable to the solution of torsional vibration problems. In all other cases, more than one mass and, in most cases, more than one connecting shaft, are involved, so that the extension of the principles already considered will now be applied to multi-mass systems.

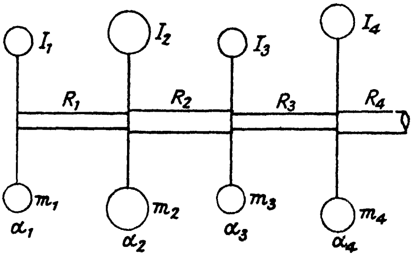

Fig. 5.—System Comprising ($n + 1$) Masses and n Shafts.

Referring to Equation (1) it will be seen that the natural frequency of the torsional pendulum was found by setting down the differential equation of motion of the system and obtaining its solution. From this equation the natural frequency was found directly (Equation 6).

To turn now to the system shown in Fig. 5. Damping will be neglected as experience shows that the amount of damping encountered in diesel engines is of such magnitude that the natural frequency is unaffected thereby. The system comprises $(n + 1)$ masses and n connecting shafts. I_1, I_2, etc., are the moments of inertia of the masses and R_1, R_2, etc., are the shaft stiffnesses respectively. Let α_1, α_2, etc., be the instantaneous angles of twist of the masses.

Then the following equations of motion hold good :—

$$\begin{aligned}
&I_1\ddot{\alpha}_1 + R_1(\alpha_1 - \alpha_2) &&= 0\\
&I_2\ddot{\alpha}_2 + R_2(\alpha_2 - \alpha_3) - R_1(\alpha_1 - \alpha_2) &&= 0\\
&I_3\ddot{\alpha}_3 + R_3(\alpha_3 - \alpha_4) - R_2(\alpha_2 - \alpha_3) &&= 0\\
&\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot &&\cdot\\
&\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot &&\cdot\\
&I_n\ddot{\alpha}_n + R_n(\alpha_n - \alpha_{n+1}) - R_{n-1}(\alpha_{n-1} - \alpha_n) &&= 0\\
&I_{n+1}\ddot{\alpha}_{n+1} - R_n(\alpha_n - \alpha_{n+1}) &&= 0 \quad (22)
\end{aligned}$$

From this set of differential equations, a general equation for the natural frequency can be obtained.* By writing Equations (22) in a slightly different form, the same equation can be obtained by an alternative method.† It is not proposed to repeat these proofs here, but merely to state the equation, which is as follows :—

$$\begin{aligned}
F^n &- F^{n-1}(j_1 + j_2 + j_3 + j_4 + \dots j_{2n})\\
&+ F^{n-2}\{j_1(j_3 + j_4 + \dots j_{2n}) + j_2(j_4 + \dots j_{2n}) + \dots\}\\
&- F^{n-3}[j_1\{j_3(j_5 \dots j_{2n}) + j_4(j_6 \dots j_{2n}) + \dots\}\\
&\qquad + j_2\{j_4(j_6 \dots j_{2n}) + \dots\} + j_3\{\dots\} + \dots]\\
&+ F^{n-4}[\dots] - \dots\dots = 0 \quad . \quad . \quad . \quad (23)
\end{aligned}$$

where F represents the natural frequency (squared) of the whole system, and j_1, j_2, etc., are the individual frequencies (squared), *i.e.*

---

* Lochner, *Jl.I.E.E.*, 1927, Vol. 65, p. 76.

† Bradbury, *Proc.I.Mech.E.*, 1933, Vol. 125, p. 483.

j_1 is the frequency (squared) of mass m_1 on shaft 1
j_2 ,, ,, ,, ,, m_2 ,, 1
j_3 ,, ,, ,, ,, m_2 ,, 2
j_4 ,, ,, ,, ,, m_3 ,, 2
etc.

The equation is built up as follows. The first term contains F to the nth power, where n is the number of connecting shafts. The second term contains F to the power $(n - 1)$ multiplied by the sum of the individual frequencies (squared). The third term contains F to the power $(n - 2)$ multiplied by the sum of the individual frequencies (squared) up to j_{2n-2}, each of which is first multiplied by the sum from j_{x+2} to j_{2n}, and so on.

The equation has n roots each of which represents the square of a natural frequency of the complete system. Thus the important fact is established that a system such as that shown in Fig. 5 has possible natural frequencies equal in number to the connecting shafts.

Now consider the case where $n = 1$. This gives the simple two mass-one shaft system shown in Fig. 6. Putting $n = 1$ in Equation (23) we have

$$F - (j_1 + j_2) = 0$$

whence

$$F = j_1 + j_2 \quad . \quad . \quad . \quad (24)$$

Substituting $j_1 = \left(\frac{1}{2\pi}\right)^2 \frac{R_1}{I_1}$, $j_2 = \left(\frac{1}{2\pi}\right)^2 \frac{R_1}{I_2}$

$$F = \left(\frac{1}{2\pi}\right)^2 \left\{\frac{R_1}{I_1} + \frac{R_1}{I_2}\right\}$$

whence the natural frequency

$$f = \frac{1}{2\pi}\sqrt{R_1\left(\frac{1}{I_1} + \frac{1}{I_2}\right)} \quad . \quad . \quad (25)$$

The form or " mode " of vibration of this system is shown by the dotted line in Fig. 6. When the mass I_1 is swinging in one direction, the mass I_2 is swinging in the opposite direction, and there is therefore a point in the shaft at A where the vibrations change phase and are of zero magnitude. This point is known as a " node."

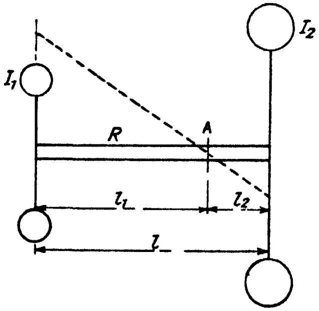

Fig. 6.—Two Mass-One Shaft System.

The position of the node is found as follows :—The system of Fig. 6 is virtually two torsional pendulums of equal frequency joined back to back, the point of juncture being the node. Letting the total length of the shaft be l and the lengths on either side of the node l_1 and l_2 respectively, we have

$$f = \frac{1}{2\pi}\sqrt{\frac{Rl}{I_1 l_1}} = \frac{1}{2\pi}\sqrt{\frac{Rl}{I_2 l_2}}$$

whence $$I_1 l_1 = I_2 l_2$$

and $$\frac{I_1}{I_2} = \frac{l_2}{l_1}$$

Thus the shaft is divided in inverse proportion to the moments of inertia of the masses.

EXAMPLE.—*Two Mass-One Shaft System.*

The system which is shown in Fig. 6 has the following specification :—

$$I_1 = 36{\cdot}3 \text{ lbs. ft.}^2 \text{ sec.}^2$$
$$I_2 = 100 \text{ lbs. ft.}^2 \text{ sec.}^2$$
$$R = 4{\cdot}7 \times 10^6 \text{ ft. lbs. per radian.}$$
$$l = 10 \text{ inches.}$$

By Equation (24)

$$F = j_1 + j_2$$
$$j_1 = \left(\frac{1}{2\pi}\right)^2 \frac{4{\cdot}7 \times 10^6}{36{\cdot}3} = 3280$$
$$j_2 = \left(\frac{1}{2\pi}\right)^2 \frac{4{\cdot}7 \times 10^6}{100} = 1190$$
$$F = j_1 + j_2 = 4470$$

Natural frequency

$$f = \sqrt{4470} = 66{\cdot}8 \text{ v.p.s.}$$

Or by Equation (25)

$$f = \frac{1}{2\pi}\sqrt{\frac{4{\cdot}7 \times 10^6 \times 136{\cdot}3}{100 \times 36{\cdot}3}}$$
$$= 66{\cdot}8 \text{ v.p.s.}$$

To find the position of the node

$$\frac{l_2}{l_1} = \frac{I_1}{I_2} = 0{\cdot}363$$

whence $l_2 = 0{\cdot}363 l_1$

and since $l = l_1 + l_2 = 10$ ins.

$$l_1 = \frac{10}{1{\cdot}363} = 7{\cdot}35 \text{ ins.}$$

Now consider the three mass-two shaft system shown in Fig. 7. Putting $n = 2$ in Equation (23) we have

$$F^2 - F(j_1 + j_2 + j_3 + j_4) + [j_1(j_3 + j_4) + j_2 j_4] = 0 \quad (26)$$

giving two values for F and hence two natural frequencies.

The lower of these two natural frequencies will occur with one of the end masses swinging against the other two masses, the node lying in one of the two connecting shafts. The higher natural frequency will occur with the two end masses swinging in phase, the centre mass swinging opposite in phase. In this case the nodes will lie one in each connecting shaft.

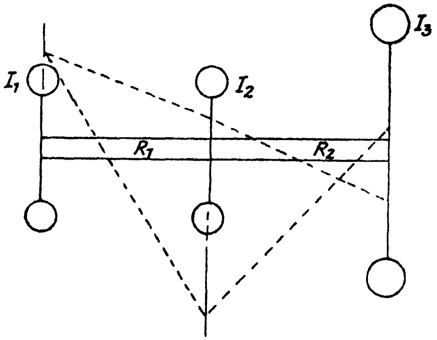

Fig. 7.—Three Mass-Two Shaft System.

These two modes of vibration are shown by the dotted lines in Fig. 7.

The modes of vibration again depend on the relative magnitudes of the masses and shaft stiffnesses. In general it may be assumed that if $(j_1 + j_2)$ is less than $(j_3 + j_4)$, the node for the lower frequency will lie in the first shaft and *vice versa.* Also if $I_1 = I_3$ and

$R_1 = R_2$, the node will lie in the centre mass. The determination of the actual positions of the nodes is best carried out by the method described in the following section on calculation by successive approximation.

EXAMPLE.—*Three Mass-Two Shaft System.*

The system which is shown in Fig. 7 has the following specification :—

$I_1 = 36$ lbs. ft.^{2} sec.^{2}
$I_2 = 36$ lbs. ft.^{2} sec.^{2}
$I_3 = 100$ lbs. ft.^{2} sec.^{2}
$R_1 = 5{\cdot}02 \times 10^6$ ft. lbs. per radian.
$R_2 = 5{\cdot}38 \times 10^6$ ft. lbs. per radian.

By Equation (26)

$$F^2 - F(j_1 + j_2 + j_3 + j_4) + [j_1(j_3 + j_4) + j_2 j_4] = 0$$

The insertion of the appropriate values in this equation is best carried out in tabular form, as follows :—

	I	R	$\frac{R}{I}$	$\left(\frac{1}{2\pi}\right)^2 \frac{R}{I}$	$(j_3 + j_4)$	j_4
j_1	36	$5{\cdot}02 \times 10^6$	$13{\cdot}9 \times 10^4$	3520	...	...
j_2	36	$5{\cdot}02 \times 10^6$	$13{\cdot}9 \times 10^4$	3520	...	...
j_3	36	$5{\cdot}38 \times 10^6$	$14{\cdot}9 \times 10^4$	3770	3770	...
j_4	100	$5{\cdot}38 \times 10^6$	$5{\cdot}38 \times 10^4$	1360	1360	1360
...	...	...	...	12170	5130	1360

$$F^2 - F(12170) + 3520(5130) + 3520(1360) = 0$$
$$F^2 - F(12{\cdot}17 \times 10^3) + 22{\cdot}8 \times 10^6 = 0$$

which factorises as

$$(F - 2320)(F - 9850) = 0$$

The frequencies thus are

One node.

$$f_1 = \sqrt{2320} = 48{\cdot}2 \text{ v.p.s.}$$

Two node.

$$f_2 = \sqrt{9850} = 99{\cdot}3 \text{ v.p.s.}$$

The two modes of vibration, which give the positions of the nodes, are shown by the dotted lines. As previously mentioned, the determination of the positions of these nodes is best carried out by the method described in the following section.

Frequencies by Successive Approximation.—Whilst it is possible to evaluate the natural frequencies of a system comprising any number of masses by means of the equation already given, the method is apt to become very laborious when n is greater than 2. For this reason it is usual, when calculating systems having four or more masses to use the method of " successive approximation."

Referring again to Equations (22), these equations will have solutions of the form

$$\alpha_1 = a_1 \sin (\omega t + \varepsilon_1)$$
$$\alpha_2 = a_2 \sin (\omega t + \varepsilon_2),$$
$$\text{etc.}$$

where ω is the phase velocity of the particular frequency under consideration.

Substituting these values of α_1, α_2, etc., in Equations (22) we have:

$$\begin{aligned}
&I_1 a_1 \omega^2 - R_1(a_1 - a_2) &&= 0\\
&I_2 a_2 \omega^2 - R_2(a_2 - a_3) + R_1(a_1 - a_2) &&= 0\\
&I_3 a_3 \omega^2 - R_3(a_3 - a_4) + R_2(a_2 - a_3) &&= 0\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&I_n a_n \omega^2 - R_n(a_n - a_{n+1}) + R_{n-1}(a_{n-1} - a_n) &&= 0\\
&I_{n+1} a_{n+1} \omega^2 + R_n(a_n - a_{n+1}) &&= 0 \qquad (27)
\end{aligned}$$

For calculation purposes, these equations are more conveniently written in the following form:—

$$\begin{aligned}
a_2 &= a_1 - \frac{\omega^2}{R_1}(I_1 a_1)\\
a_3 &= a_2 - \frac{\omega^2}{R_2}(I_1 a_1 + I_2 a_2)\\
a_4 &= a_3 - \frac{\omega^2}{R_3}(I_1 a_1 + I_2 a_2 + I_3 a_3)\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
a_{n+1} &= a_n - \frac{\omega^2}{R_n}(I_1 a_1 + I_2 a_2 + \ldots I_n a_n) \quad . \quad (28)
\end{aligned}$$

Now, by making an estimate of the value of ω^2, and assuming an arbitrary value, say unity, for a_1, the angular displacement of the first mass, the value of a_2 can be calculated from the first equation. a_2 having been found, a_3 can be calculated from the second equation and so on up to the last equation, which gives the value of a_{n+1}, the angular displacement of the last mass. If ω^2 has been correctly chosen, the equation

$$I_1 a_1 \omega^2 + I_2 a_2 \omega^2 + \ldots + I_{n+1} a_{n+1} \omega^2 = 0 \qquad (29)$$

representing the sum of Equations (27), will be satisfied. If this equation is not satisfied, fresh values of ω^2 must be tried until satisfaction is obtained.

The following example shows a simple method of setting out this calculation in tabular form, the system being that of a six cylinder diesel engine as shown in Fig. 8. The specification is as follows :—

Moment of inertia per cylinder line . . . . . = 7·11 lbs. ft.^{2} sec.^{2}
Moment of inertia of flywheel = 400 lbs. ft.^{2} sec.^{2}
Stiffness between cylinder centres = $14{\cdot}1 \times 10^6$ ft. lbs. per radian.
Stiffness between No. 6 cylinder and flywheel . . . = $14{\cdot}25 \times 10^6$ ft. lbs. per radian.

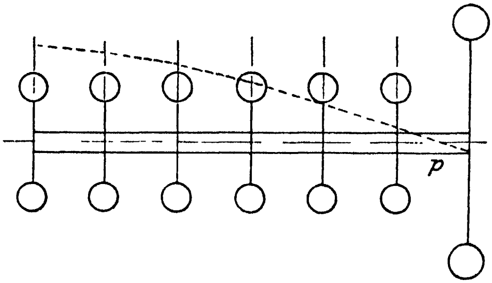

Fig. 8.—Six Cylinder Diesel Engine.

Column 1 gives the moments of inertia of the masses in ft. lb. second units, while Column 2 gives the moment of inertia multiplied by the square of the angular velocity ω. Column 3 gives the angles of rotation of the masses, that of mass No. 1 being given an arbitrary value of unity. Column 4 gives the moments of the inertia forces of the masses and Column 5 the total

(Table overpage)

TABLE I

	$\omega = 355$				$\omega^2 = 126025$			
	1	2	3	4	5	6	7	8
Mass No.	I	$I\omega^2$	a	$I\omega^2 a$	$\Sigma I\omega^2 a$	R	$\frac{\Sigma I\omega^2 a}{R}$	Stress lbs./sq. in.
1.	7·11	$\cdot896\times10^6$	1·0000	$\cdot896\times10^6$	$\cdot896\times10^6$	$14\cdot1\times10^6$	·0636	1850
2.	7·11	$\cdot896\times10^6$	·9364	$\cdot839\times10^6$	$1\cdot735\times10^6$	$14\cdot1\times10^6$	·1230	3580
3.	7·11	$\cdot896\times10^6$	·8134	$\cdot730\times10^6$	$2\ 465\times10^6$	$14\cdot1\times10^6$	·1748	5100
4.	7·11	$\cdot896\times10^6$	·6386	$\cdot572\times10^6$	$3\cdot037\times10^6$	$14\cdot1\times10^6$	·2154	6280
5.	7·11	$\cdot896\times10^6$	·4232	$\cdot379\times10^6$	$3\cdot416\times10^6$	$14\cdot1\times10^6$	·2423	7070
6.	7·11	$\cdot896\times10^6$	·1809	$\cdot162\times10^6$	$3\cdot578\times10^6$	$14\cdot25\times10^6$	·2511	7410
7.	400	$50\cdot4\times10^6$	$-\cdot0702$	$-3\cdot54\times10^6$	$\cdot038\times10^6$	...	...	...

twisting moment of the inertia forces of all the masses to the left of the section under consideration. The shaft stiffnesses are given in Column 6 and these, divided into the twisting moments given in Column 5, give the angles of twist for consecutive parts of the shaft (Column 7).

To satisfy Equation (29) the final figure in Column 5 should be zero. To obtain this exactitude, however, is very laborious, and it will be found that where the final error is of the order of that in the above example, namely 1 per cent., the results obtained are sufficiently accurate for all practical purposes.

The natural frequency now is

$$f = \frac{\omega}{2\pi} = \frac{355}{2\pi} = 56{\cdot}5 \text{ v.p.s.}$$

The values of a, given in Column 3, when plotted as shown in Fig. 8, give the "normal elastic curve" of the system. This curve shows the relative amplitudes of the various parts of the system for free vibration, and these amplitudes remain in the proportion given, whatever the actual value of a may be at mass No. 1.

The point p (Fig. 8) where the normal elastic curve crosses the base line is the node of the system. Thus this method gives the positions of the nodes as an incidental part of the frequency calculation.

In Column 8 are given the stresses in each section of the shafting due to unit deflection at No. 1 crank, the shaft diameter being 8 inches. These values, which are filled in after the frequency calculation is completed, are obtained from the relationship

$$S = \frac{T}{Z_T}$$

where S = torsional stress in lbs. per sq. in.

T = torque on section in lbs. inches.

Z_T = section modulus.

$= \frac{\pi d^3}{16}$ (inch^3 units), d being the shaft diameter in inches.

The total torque on each consecutive section in ft. lbs. for one radian twist at No. 1 crank is given by Column 5, so that Column 8 is calculated from

$$S = \frac{12 \text{ (Value in Column 5)}}{57 \cdot 3 \pi d^3 / 16}$$

$$= \frac{1 \cdot 06 \text{ (Column 5)}}{d^3} \text{ lbs. per sq. in. per degree.}$$

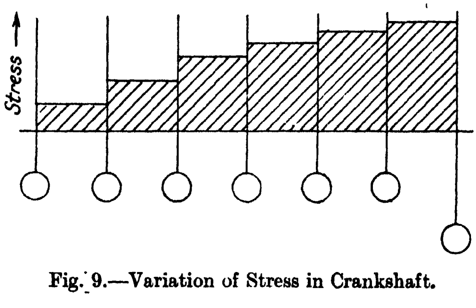

Fig. 9.—Variation of Stress in Crankshaft.

The various stresses are shown diagrammatically in Fig. 9.

If there is a variation in shaft diameter at any section of the shaft, say from d to d_1, the stress per degree at that section will be altered in the ratio $\left(d/d_1\right)^3$

Gorfinkel's Method.—A modified form of the above calculation, in which the work involved is reduced by means of the use of non-dimensional coefficients, has been devised by Gorfinkel.* Referring to the last of Equations (28)

$$a_{n+1} = a_n - \frac{\omega^2}{R_n}(I_1a_1 + I_2a_2 + \ldots I_na_n)$$

$$= a_n - \omega^2 . \frac{1}{R_n}\sum_1^n I_na_n$$

Letting $\varrho_n = {}^{I_n}\!/_{I_1}$, $\lambda_n = {}^{l_n}\!/_{l_1} = {}^{R_1}\!/_{R_n}$ where l is the "equivalent length" of shaft used in calculating the stiffness R, $\alpha_n = {}^{a_n}\!/_{a_1}$ and $X^2 = {}^{\omega^2 I_1}\!/_{R_1}$, we have

$$\alpha_{n+1}\, a_1 = \alpha_n a_1 - \frac{X^2R_1}{I_1} . \frac{\lambda_n}{R_1}\sum_1^n \varrho_n I_1\alpha_n a_1$$

$$\alpha_{n+1} = \alpha_n - X^2\, \lambda_n \sum_1^n \varrho_n\alpha_n \quad . \quad . \quad . \quad . \quad (30)$$

Assuming any value for X^2, successive values of α are determined, and if X^2 has been correctly chosen, $\sum_1^{n+1}\varrho\alpha$ will equal zero.

Remembering that $X^2 = \omega^2 I_1/R_1$ and that the natural frequency is given by $\omega/2\pi$, we have

$$\text{Natural frequency} = \frac{X}{2\pi}\sqrt{\frac{R_1}{I_1}} \text{ v.p.s.} \quad . \quad . \quad (31)$$

Again in this case, the calculation is most conveniently arranged in tabular form. In what follows, the previous example has been worked out by this method.

**Engineering*, 27th December, 1929, p. 827.

TABLE II

$X^2 = \cdot 0636$

Mass No.	ρ	a	$\Sigma\rho a$	λ	$X^2\lambda\Sigma\rho a$
1.	1	1·0000	1·0000		·0636
		·0636	·9364	1	
2.	1	9364	1·9364		·1230
		·1230	·8134	1	
3.	1	·8134	2·7498		·1748
		·1748	·6386	1	
4.	1	·6386	3·3884		·2154
		·2154	·4232	1	
5.	1	·4232	3·8116		·2423
		·2423	·1809	1	
6.	1	·1809	3·9925		·2511
		·2511	−3·9452	·99	
7.	56·2	−·0702	·0473		

The natural frequency is

$$f = \frac{X}{2\pi}\sqrt{\frac{R_1}{I_1}} = \frac{1}{2\pi}\sqrt{\frac{0{\cdot}0636 \times 14{\cdot}1 \times 10^6}{7{\cdot}11}} = 56{\cdot}5 \text{ v.p.s.}$$

It will be noted that the ordinates of the normal elastic curve are again given by this method, namely, by the third column in the table.

Approximate Frequency Calculations.—The methods of calculation described in the two preceding sections must be used when the natural frequency is required to a high degree of accuracy and when, for the calculation of stresses and amplitudes, the normal elastic curve is required.

It occasionally happens, however, particularly in the preliminary stages of design, that an approximate figure for the natural frequency is required. In these cases it is possible to use abbreviated methods of calculation which will give the one node natural frequency to within about 5 per cent.

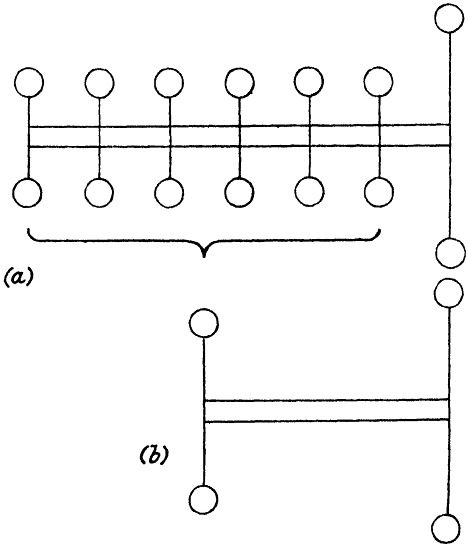

Fig. 10.—Six Cylinder Engine and Equivalent Two-Mass System.

(*a*) *The Concentrated Mass Method.*—Referring to Fig. 10 (*a*), which is a reproduction of Fig. 8, if the whole of masses 1 to 6 is concentrated at the centre line of the engine, a simple two mass-one shaft system is obtained. This is shown in Fig. 10 (*b*). The combined moment of inertia now equals 42·66 lbs. ft.^{2} sec.^{2}, the equivalent length of shaft is 8·14 ft., and the stiffness $4{\cdot}05 \times 10^6$ lbs. ft. per radian.

This gives a natural frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{4{\cdot}05 \times 10^6 \times 442{\cdot}66}{400 \times 42{\cdot}66}} = 51{\cdot}6 \text{ v.p.s.}$$

as against 56·5 v.p.s. for the exact calculation, the error being 8·7 per cent.

It will be seen from the above that to group the whole of masses 1 to 6 at the engine centre line gives a frequency which is too low. A much closer approximation is obtained by concentrating a certain percentage of the masses at the engine centre line. This percentage, which varies with the number of cylinders, is given in the following table :—

Number of cylinders, . .	1	2	3	4	5	6	7	8
Percentage of Total Masses,	100	90	86·5	85	85	85	85	85

Thus in the example under consideration, the grouped mass now becomes $0{\cdot}85 \times 42{\cdot}66$ lbs. ft.^{2} sec.^{2} and the natural frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{4{\cdot}05 \times 10^6 \times 436{\cdot}3}{400 \times 36{\cdot}3}} = 55{\cdot}5 \text{ v.p.s.}$$

the error now being 1·77 per cent.

(*b*) *The Graphical Method.*—The degree of accuracy of the graphical method depends on three conditions, all of which are fortunately fulfilled by the majority of present-day diesel engines. These conditions are :—

(i) The moments of inertia per cylinder are equal.

(ii) The ratio $\dfrac{\text{moment of inertia of flywheel}}{\text{moment of inertia per cylinder}}\left(= {}^{I_F}\!\big/_{I_1}\right)$ is not less than 40.

(iii) The equivalent lengths of shaft between cylinder centres are equal.

Fig. 8 is a typical diesel engine system which fulfils

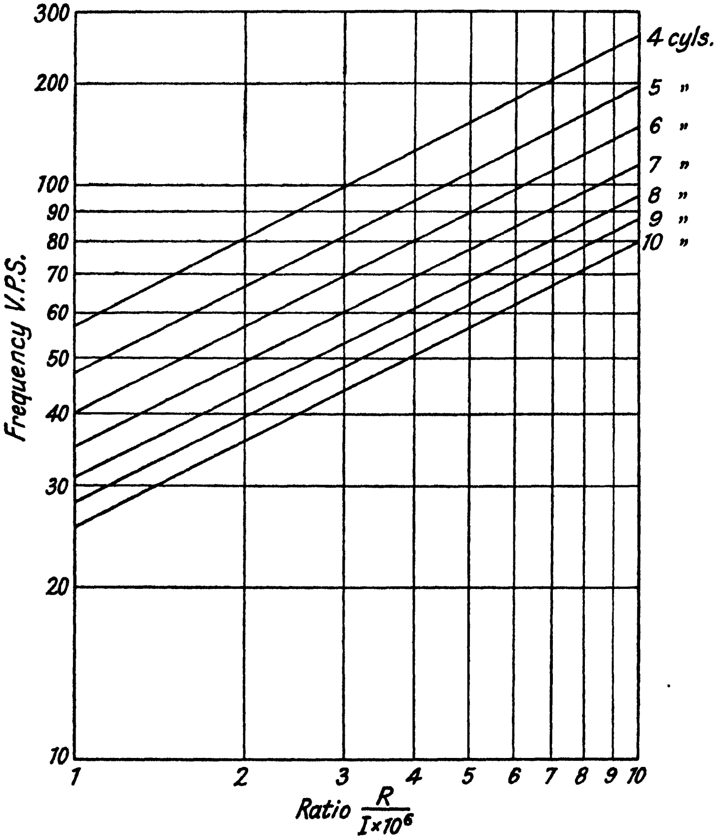

Fig. 11.—Natural Frequency Graph.

these conditions and Figs. 11 and 12 are the graphs from which the natural frequency may be read direct.

In Fig. 11 the natural frequency f is plotted against the ratio $R/I \times 10^6$ for engines having from 4 to 10 cylinders. The ratio l_F/l_1 is assumed to be unity and the ratio I_F/I_1 is taken as 60, this being a good average value for present-day diesel engine practice. (l_F is the equivalent length of shaft between the last cylinder and the flywheel, l_1 is the equivalent length between cylinder centres.)

Since the graph is a series of straight lines when plotted logarithmically, the natural frequency may be expressed as a single equation in terms of the gradient and the intercept on the f axis. The gradient is, as would be anticipated, 0·5 and the equation therefore becomes

$$f = \frac{U}{10^3}\sqrt{\frac{R_1}{I_1}} \text{ v.p.s.}$$

where U has the following values :—

10 cylinders . . .	$U = 25{\cdot}45$
9 cylinders	27·9
8 cylinders	31·05
7 cylinders	34·9
6 cylinders	40
5 cylinders	46·9
4 cylinders	56·9

Fig. 12 is a supplementary graph to take into account variations in the ratio l_F/l_1. In this graph the multiplying factor V has been plotted against the ratio l_F/l_1 for values of the latter from 0·8 to 1·2. Thus the natural frequency of any diesel engine complying with conditions (i), (ii) and (iii) above is given by

$$f = \frac{UV}{10^3}\sqrt{\frac{R_1}{I_1}} \text{ v.p.s.} \quad . \quad . \qquad (32)$$

where U has the values already given and V is read from Fig. 12 for the particular number of cylinders and l_F/l_1 ratio under consideration.

The Introduction of Gearing.—The introduction of gearing into a torsionally elastic system renders the problem of torsional vibration more complex than that of the normal straight system. The ratio of the gears must be allowed for, whilst if two equal systems are mutually geared to a third, two separate and distinct series of frequencies are possible. This is made clear by examination of the following simple systems.

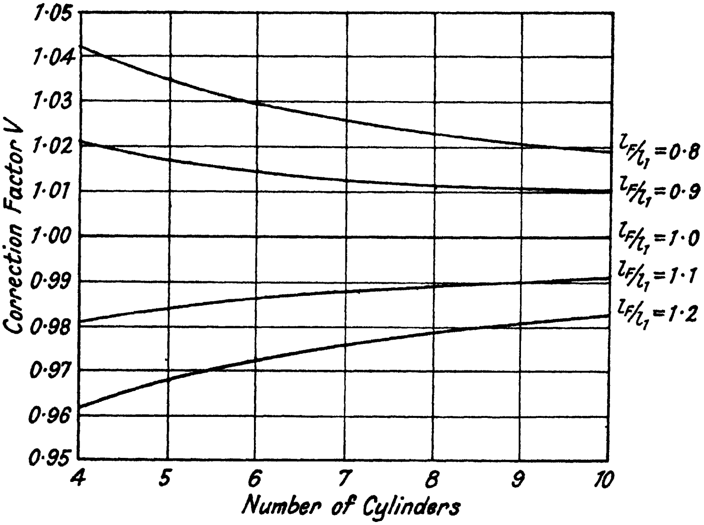

Fig. 12.—Graph of Correction Factors.

Fig. 13 (*a*) shows two masses and shafts mutually geared together. The system is assumed to be

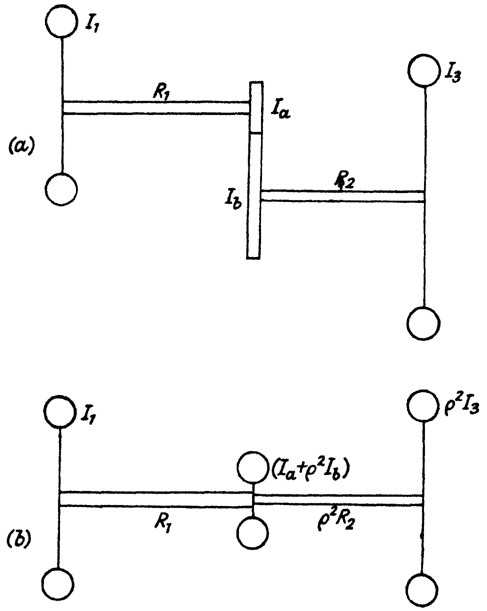

Fig. 13.—Single Geared System.

constrained except for torsional movement. The nomenclature is as follows :—

I_1 = moment of inertia of first mass.
I_a = moment of inertia of small gear.
I_b = moment of inertia of large gear.
I_3 = moment of inertia of second mass.
R_1 = stiffness between first mass and gears.
R_2 = stiffness between gears and second mass.
ϱ = gear ratio, *i.e.* ratio of smaller to larger gear.

It can be shown[*] that the natural frequencies of this system are given by the equation

$$F^2 - F(j_1 + j_2 + \varrho^2 j_3 + j_4) + j_1(\varrho^2 j_3 + j_4) + j_2 j_4 = 0 \qquad (33)$$

---

[*] Bradbury, *Proc.I.Mech.E.*, 1933, Vol. 125, p. 485 *et seq.*

The notation is the same as that for Equation (23) and the combined moment of inertia of the gears is $(I_a + \varrho^2 I_b)$. Thus it is seen that the system may be represented by an equation of the form of Equation (23) by putting $n = 2$ and multiplying the moments of inertia and the stiffness beyond the first gear by the square of the gear ratio. The equivalent system, in which all masses rotate at the same speed, is shown in Fig. 13 (*b*).

The system shown in Fig. 14 (*a*) is the same as that of Fig. 13 (*a*), except that the first mass, shaft and gear are duplicated.

It can be shown* that the natural frequencies of this system are given by the equation

$$\{F - j_1\}\{F^2 - F(j_1 + 2j_2 + \varrho^2 j_3 + j_4) + j_1(\varrho^2 j_3 + j_4) + 2j_2 j_4\} = 0 \qquad (33a)$$

The notation is again the same as that for Equation (23) and the combined moment of inertia of the gears is $(2I_a + \varrho^2 I_b)$.

The second bracket of Equation (33*a*) proves that the system of Fig. 14 (*a*) may be represented by an equation of the form of Equation (23) provided that the first mass, shaft stiffness and gear mass are doubled, and each subsequent mass and shaft stiffness is multiplied by the square of the gear ratio. In addition the first bracket shows that there is a frequency due to the two equal masses swinging one against the other, the points of juncture with the gears being the nodes. The equivalent systems in which all masses rotate at the same speed are shown in Fig. 14 (*b*) and (*c*).

These two examples serve to illustrate the type of vibration to be expected. In the geared diesel engine drive, however, the number of masses and shafts is so great that it is both quicker and more advantageous

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* Bradbury, *Proc.I.Mech.E.*, 1933, Vol. 125, p. 485 *et seq.*

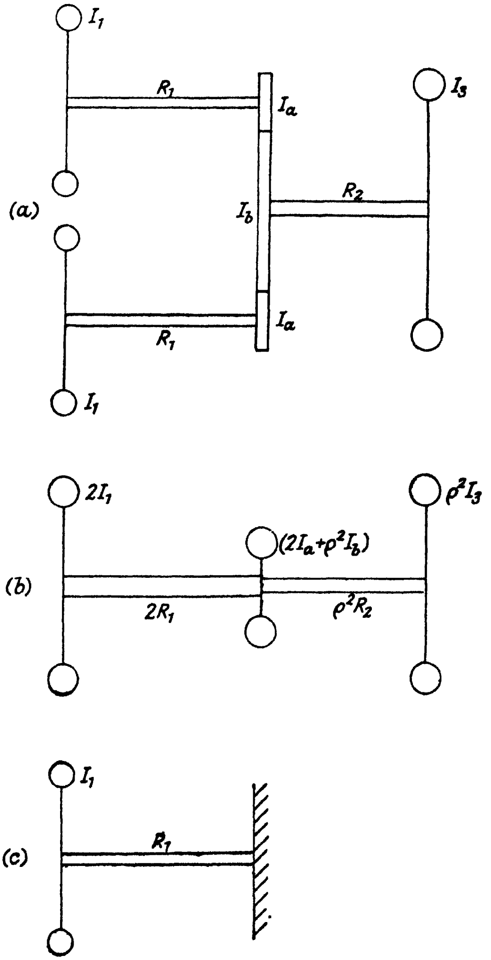

Fig. 14.—Double Geared System.

to calculate the frequencies by successive approximation.

This method of calculation is carried out in the manner already described in Table I and the calculation proceeds in the normal manner until the gears are reached. At this point, the angle of twist (Column 3) is multiplied by the gear ratio and the total torque (Column 5) is divided by the gear ratio. The calculation then proceeds as before. The normal elastic curve and the total torque on each consecutive section of shaft per radian twist at No. 1 mass are again obtained as an incidental part of the frequency calculation.

The following table gives the calculation for the one node frequency of the geared marine diesel engine system shown in Fig. 15. The specification of the system is as follows:—

Moment of inertia per cylinder line . . . . . = 18·73 lbs. ft.^{2} sec.^{2}

Moment of inertia of flywheel = 383 lbs. ft.^{2} sec.^{2}

Moment of inertia of first gear . = 2 lbs. ft.^{2} sec.^{2}

Moment of inertia of second gear = 42 lbs. ft.^{2} sec.^{2}

Moment of inertia of propeller (including entrained water) . = 174 lbs. ft.^{2} sec.^{2}

Stiffness between cylinder centres = 18·12 × 10^6 ft. lbs. per radian

Stiffness between No. 7 cylinder and flywheel . . . = 18·12 × 10^6 ft. lbs. per radian.

Stiffness between flywheel and first gear . . . . = 0·315 × 10^6 ft. lbs. per radian.

Stiffness between second gear and propeller . . . = 1·005 × 10^6 ft. lbs. per radian.

Reduction gear ratio . . = 1/2·5

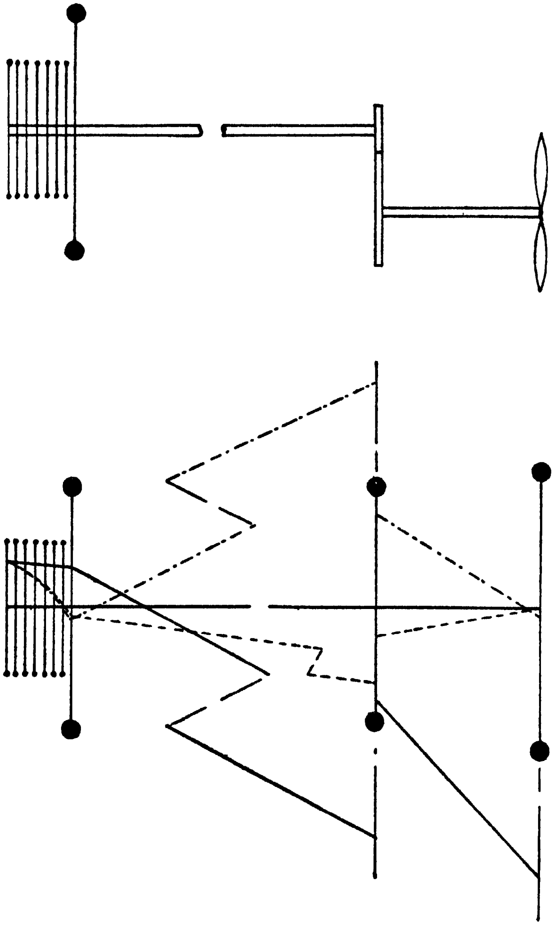

Fig. 15.—Seven Cylinder Marine Diesel Engine with Reduction Gearing.

TABLE III

	$\omega = 62\cdot1$				$\omega^2 = 3860$		
	1	2	3	4	5	6	7
Mass No.	I	$I\omega^2$	a	$I\omega^2 a$	$\Sigma I\omega^2 a$	R	$\frac{\Sigma I\omega^2 a}{R}$
1.	18·73	$\cdot0723\times10^6$	1·0000	$\cdot0723\times10^6$	$\cdot0723\times10^6$	$18\cdot12\times10^6$	·0040
2.	18·73	$\cdot0723\times10^6$	·9960	$\cdot0720\times10^6$	$\cdot1443\times10^6$	$18\cdot12\times10^6$	·0080
3.	18·73	$\cdot0723\times10^6$	·9880	$\cdot0715\times10^6$	$\cdot2158\times10^6$	$18\cdot12\times10^6$	·0119
4.	18·73	$\cdot0723\times10^6$	·9761	$\cdot0705\times10^6$	$\cdot2863\times10^6$	$18\cdot12\times10^6$	·0158
5.	18·73	$\cdot0723\times10^6$	·9603	$\cdot0694\times10^6$	$\cdot3557\times10^6$	$18\cdot12\times10^6$	·0196
6.	18·73	$\cdot0723\times10^6$	·9407	$\cdot0680\times10^6$	$\cdot4237\times10^6$	$18\cdot12\times10^6$	·0234
7.	18·73	$\cdot0723\times10^6$	·9173	$\cdot0663\times10^6$	$\cdot4899\times10^6$	$18\cdot12\times10^6$	·0271
Flywheel . . .	383	$1\cdot478\times10^6$	·8902	$1\cdot3157\times10^6$	$1\cdot8056\times10^6$	$\cdot315\times10^6$	5·7321
First Gear . .	2	$\cdot0077\times10^6$	$-4\cdot8419$ $(\div 2\frac{1}{2})$	$-\cdot0373\times10^6$	$1\cdot7683\times10^6$ $(\times 2\frac{1}{2})$ $4\cdot4208\times10^6$	...	...
Second Gear . .	42	$\cdot1620\times10^6$	$-1\cdot9368$	$-\cdot3138\times10^6$	$4\cdot1070\times10^6$	$1\cdot005\times10^6$	4·0865
Propeller . . .	174	$\cdot6720\times10^6$	$-6\cdot0231$	$-4\cdot0475\times10^6$	$\cdot0595\times10^6$	...	...

The remainder in Column 5 shows an error of approximately one per cent., so the value of $\omega = 62{\cdot}1$ may be taken as correct.

The natural frequency thus is

$$f = \omega/2\pi = 9{\cdot}9 \text{ v.p.s.}$$
$$= 594 \text{ v.p.m.}$$

In Fig. 15 is plotted the one node normal elastic curve as calculated in the above Table, together with the two and three node forms which are calculated in exactly the same manner.

Referring back to Fig. 14, it was stated in the case of the two node frequency that the two equal masses swing against each other, the points of juncture with the gears being the nodes. This brings out a very important point in connection with twin engines geared to a common propeller shaft.

Provided that the engines are identical in every respect and that the cranks are absolutely in phase (this is the usual arrangement for synchronised starting and reversing) the mode of vibration just described is unexcitable since the disturbing forces attempt to oscillate the crankshafts in phase and the natural mode of vibration is exactly out of phase.

Thus with this type of installation, torsional vibration with an even number of nodes cannot appear in practice, and only those with an odd number of nodes need be considered.

EQUIVALENT DYNAMICAL SYSTEMS

In all examples discussed so far, what are known as "ideal" systems have been considered. That is, the masses have been assumed to be uniform circular rings, and the shafts weightless and uniform in diameter. In actual practice much more complex conditions prevail, so that it is necessary, before calculating an actual installation, to replace it by an equivalent "ideal" system. Fig. 16 shows an actual dynamical system comprising a six cylinder diesel engine direct coupled to an alternator, together with its equivalent "ideal" system.

(*a*) **Equivalent Lengths of Shafts.**—By equivalent length of shaft is meant that length of uniform shaft which, when subjected to a given torque, will twist through an angle equal to that produced by the same torque on the variable section shaft.

(i) *The Crankshaft.*—The engine crankshaft is a very complicated member, the equivalent length of which cannot be accurately calculated from first principles. As a result, however, of some very valuable work by Major B. C. Carter, an empirical formula* has been devised which is found to give very reliable results in practice. Referring to Fig. 17, the formula is as follows :—

Equivalent length from A to B in terms of journal shaft

$$= L = (2b + 0{\cdot}8h) + \frac{3(D_1^4 - d_1^4)a}{4(D_2^4 - d_2^4)} + \frac{3r}{2}\,\frac{(D_1^4 - d_1^4)}{hw^3} \quad (34)$$

---

* *Engineering*, 13th July, 1928, p. 36.

Corresponding stiffness from A to B

$$\therefore R = \frac{G\pi(D_1^4 - d_1^4)}{32L} \tag{35}$$

where $G = 17 \times 10^8$ lbs. per sq. ft. for steel.

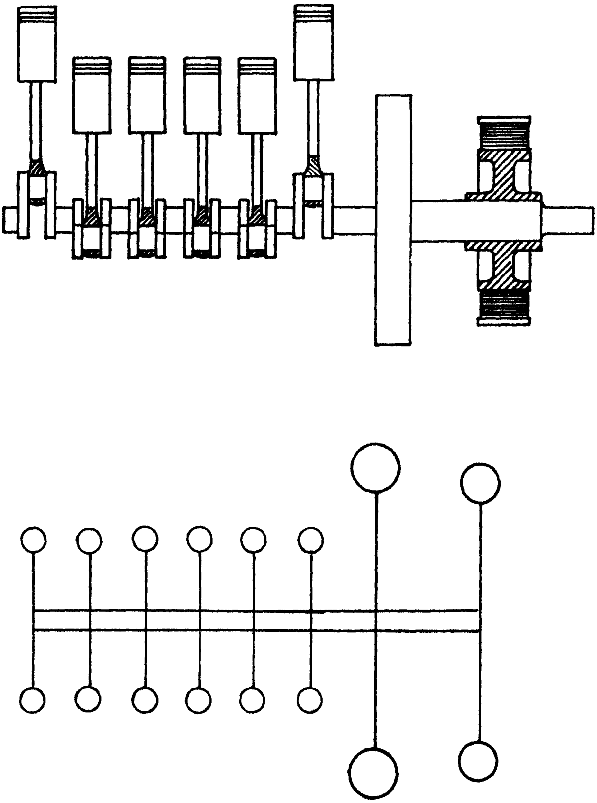

Fig. 16.—Six Cylinder Diesel Engine and Rotor and Equivalent Ideal System.

This formula was obtained by twisting actual aero, automobile and marine engine crankshafts in bearings having clearances approximating to those used in ordinary working. It will be noted that, in the first term, it has been necessary to allow for local yielding of the webs at the point of juncture with the journals and crankpins by adding the length 0·8h. Also, from the second term, it will be seen that the crankpin is in pure torsion, *i.e.* when the crank is twisted, the journals move out of the axial. Thus when a crankshaft is vibrating torsionally, the journals move to and fro across the clearance space.

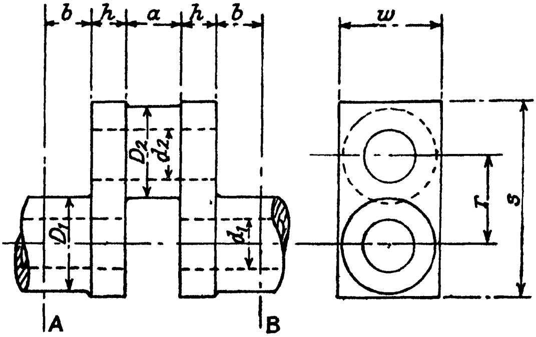

Fig. 17.—Crankshaft (Carter's Formula).

(ii) *Variable Diameter Shaft.*—Since the angle of twist produced in a given uniform shaft by a given torque is inversely proportional to the fourth power of the diameter, the equivalent length at diameter d_1 of a variable diameter shaft such as that shown in Fig. 18 is given by

$$L = l_1 + d_1^4\left(\frac{l_2}{d_2^4} + \frac{l_3}{d_3^4}\right)$$

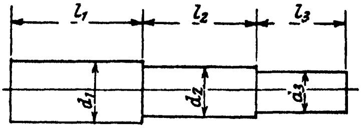

Fig. 18.—Shaft of Varying Section.

(iii) *Tapered Shaft.*—For the tapered shaft shown in Fig. 19, consider the elementary strip of width δl_1. Let the diameter at this point be $(d_1 + \beta l_1)$ where β is a constant.

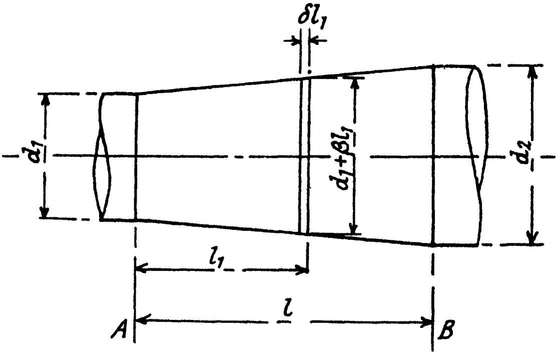

Fig. 19.—Tapered Shaft.

Then the angle of twist in the short length δl_1 is

$$\frac{C\delta l_1}{(d_1 + \beta l_1)^4}$$

where C is a constant.

Integrating, we have

Total angle of twist between A and B . . . . $= \int_0^l \frac{C}{(d_1 + \beta l_1)^4}\,\delta l_1$

$$= \frac{Cl}{3}\left(\frac{d_2^2 + d_1d_2 + d_1^2}{d_1^3d_2^3}\right)$$

since $\beta = \dfrac{d_2 - d_1}{l}$.

Now if L is the length of an equivalent shaft of diameter d_1, we have

$$\frac{CL}{d_1^4} = \frac{Cl}{3}\left(\frac{d_2^2 + d_1d_2 + d_1^2}{d_1^3d_2^3}\right)$$

whence

$$L = \frac{l}{3}\left(\frac{d_1}{d_2} + \frac{d_1^2}{d_2^2} + \frac{d_1^3}{d_2^3}\right) \quad . \quad . \quad (36)$$

When making calculations for equivalent length, it is essential that the various parts are given their correct amount of elasticity. Thus, flanged couplings should be calculated up to the bolt centres only; flanged couplings keyed on to the shaft should be calculated on the shaft diameter for half the length of the key and on the coupling diameter for the other half. Alternator rotors and generator armatures, pressed and keyed on to the shaft, should be similarly treated.

(*b*) **Equivalent Masses.**

(i) *Moment of Inertia of Flywheel.*—Fig. 20 shows a section of a typical diesel engine flywheel. It is usual, for ease of calculation, to divide the section up

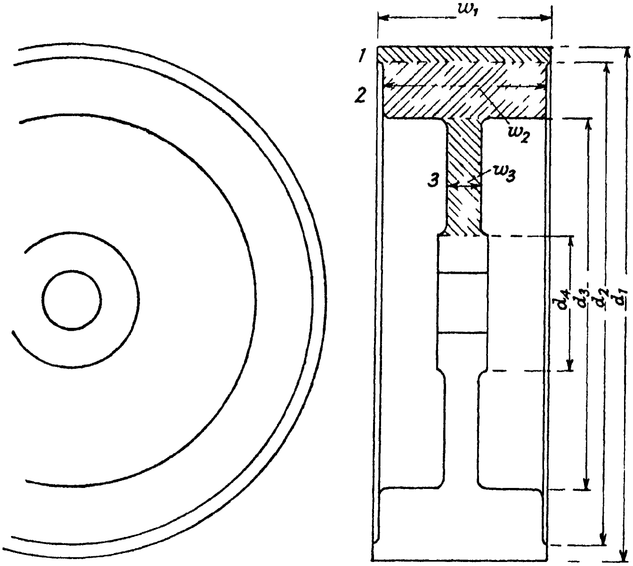

Fig 20 —Flywheel

into rectangular section rings as shown at 1, 2 and 3. Then

$$\text{Weight of Part I} = W_1 = \frac{\pi}{4}(d_1^2 - d_2^2)w_1 \times 450 \text{ lbs.}$$

for cast iron (d_1, d_2 and w_1 being measured in feet).

$$\text{Radius of gyration} = r_1 = \sqrt{\frac{d_1^2 + d_2^2}{8}} \text{ ft.}$$

$$\text{Moment of inertia of Part I} = \frac{W_1 r_1^2}{g} \text{ lbs. ft.}^2 \text{ sec.}^2$$

Thus the moment of inertia of the whole flywheel is

$$I = \frac{W_1 r_1^2 + W_2 r_2^2 + W_3 r_3^2}{g} \text{ lbs. ft.}^2 \text{ sec.}^2$$

It is usual in these calculations to neglect the flywheel effect of the hub.

(ii) *Armatures and Rotors.*—When calculating the moment of inertia of an armature or rotor there is no accurate way other than to find the Wr^2 value of each individual part and finally to add them together as in the case of the flywheel. Though this method is somewhat laborious, extremely accurate results can be obtained by careful calculation.

A method of finding the moment of inertia by experiment will be described later.

(iii) *The Engine Parts.*—Owing to the fact that the reciprocating parts do not have a constant inertia value, it is usual to substitute an equivalent revolving mass for the combined revolving and reciprocating masses of each cylinder. To do this, each part must be considered separately and a final summation made of the Wr^2 values.

The Crank Pin.—Referring to Fig. 17, the weight of the crankpin (assumed solid) is

$$W_1 = \frac{\pi}{4} \times D_2^2 \times a \times 489 \text{ lbs.}$$

The radius of gyration may, without appreciable error, be taken as the crankthrow r.

For the bore, the weight to be deducted is

$$W_2 = \frac{\pi}{4} \times d_2^2 \times (a + 2h) \times 489 \text{ lbs.}$$

The radius of gyration is again r, so that the moment of inertia of the crankpin less the bore is

$$I = \frac{(W_1 - W_2)r^2}{g} \text{ lbs. ft.}^2 \text{ sec.}^2$$

The Crankwebs.—The weight of the crankwebs (if rectangular) is given by

$$W_3 = 2hws \times 489 \text{ lbs.}$$

The radius of gyration is given by

$$r_3 = \sqrt{\frac{w^2 + s^2}{12} + \left(\frac{s}{2} - v\right)^2} \text{ ft.}$$

(where v is the distance from the centre line of the journal to the base of the crankweb)

and the moment of inertia is $\frac{W_3 r_3^2}{g}$ lbs. ft.^{2} sec.^{2}

If the crankwebs are not rectangular it will probably be more convenient to treat them in the manner given below for balance weights.

The Balance Weights.—These are best calculated by a graphical method. Referring to Fig. 21, a typical balance weight is shown with its two principal axes XX and YY, XX being the crank journal centre. The procedure for finding the moment of inertia is :—

On the right hand half of the balance weight, a rectangle OPQR is described. On to the base RQ, draw any line OA. Erect a perpendicular to cut the edge of the weight at B and C. Draw lines horizontally from B and C to cut the line OA at D and E.

Draw a number of lines similar to OA and follow the same procedure. Then by joining up the points D, E, etc., what is known as the first derived figure is obtained. This is shown in single shading on the right of Fig. 21.

The next step is to treat this first derived figure in exactly the same manner and so obtain the second derived figure shown double shaded.

Then calling the areas of the original, the first derived, and the second derived figures A_1, A_2 and A_3 respectively, the radius of gyration of the balance weight about the axis XX is given by

$$r_{xx} = \sqrt{\frac{A_3 \times (PQ)^2}{A_1}} \text{ ft.}$$

If the thickness of the weight is h and the material cast iron, the weight is given by

$$W = 2A_1h \times 450 \text{ lbs.}$$

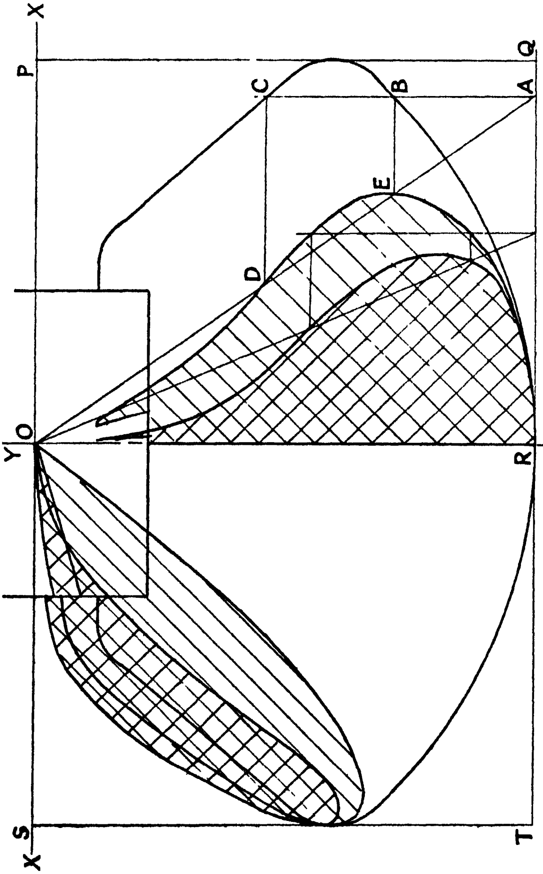

Fig. 21.—Balance Weight.

and the moment of inertia about the axis XX is

$$I_{xx} = \frac{Wr_{xx}^2}{g} \text{ lbs. ft.}^2 \text{ sec.}^2$$

It is now necessary to find the moment of inertia about the axis YY. To do this, the same procedure is followed except that ST (shown on the left of Fig. 21) is used as the base. The first and second derived figures are then as shown. Using the symbols A_1, A_2^1 and A_3^1 in the same manner as above, the radius of gyration about the axis YY is given by

$$r_{YY} = \sqrt{\frac{A_3^1 \times (OS)^2}{A_1}} \text{ ft.}$$

and the moment of inertia by

$$I_{YY} = \frac{W r_{YY}^2}{g} \text{ lbs. ft.}^2 \text{ sec.}^2$$

The rotational moment of inertia about the axis perpendicular to the paper through the point O, *i.e.* the crankshaft centre line, then is

$$I_{\text{rotational}} = I_{XX} + I_{YY} \text{ lbs. ft.}^2 \text{ sec.}^2$$

The Connecting Rod is neither wholly revolving nor wholly reciprocating, but a very close approximation to its true value can be obtained by splitting it up into its component revolving and reciprocating parts. This is usually done by the two scale method shown in Fig. 22.

The rod is carefully balanced in a horizontal position on two scale pans, knife edges being applied to the large end and small end centre lines. The weight W_1, registered by the large end scale is taken as revolving, and is added to the crankpin at the latter's radius of gyration. The weight W_2, registered by the small end scale pan, is taken as reciprocating and is added to the piston and gudgeon.

The Piston and Gudgeon, which are wholly reciprocating, must, together with the reciprocating component of the connecting rod, be replaced by an equivalent weight at crankpin radius. Referring to Fig. 23, let

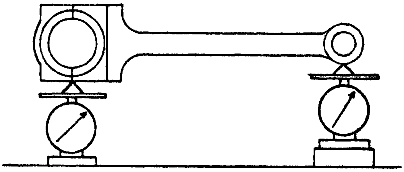

Fig. 22.—Connecting Rod Weighed by Two Scale Method.

the total reciprocating weight be W_c and the equivalent weight W. Then if the crank is revolving with an angular velocity ω radians per second, the kinetic energy of the weight W is

$$E_w = \frac{Wr^2\omega^2}{2g} \text{ ft. lbs.}$$

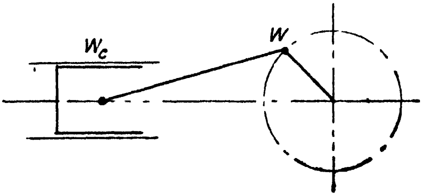

Fig. 23.—Equivalent Revolving Weight.

Now neglecting connecting rod obliquity—which entails very little error for connecting rods more than four cranks long—the velocity of the weight W_c in the crank position shown is

$$v = \omega r \sin \omega t \text{ f.p.s.}$$

where t is the time in seconds.

Thus its instantaneous kinetic energy is

$$\frac{W_c(\omega r \sin \omega t)^2}{2g} \text{ ft. lbs.}$$

and the average kinetic energy, taken over one revolution, is

$$\begin{aligned} E_c &= \frac{1}{2\pi} \times \frac{W_c\omega^2 r^2}{2g}\int_0^{2\pi} \sin^2 \omega t \, dt \\ &= \frac{W_c\omega^2 r^2}{4g} \end{aligned}$$

Thus if $E_c = E_w$, then $W = \frac{W_c}{2}$. Therefore one half of the reciprocating weight should be added to the purely revolving weight at crankpin radius.

To summarise, the masses concentrated at each cylinder centre line are :—

The crankpin (less bore, if any) at crankpin radius.
The crankwebs, acting at their radius of gyration.
The balance weights (if any) acting at their radius of gyration.
The revolving portion of the connecting rod, at crankpin radius.
One half of the reciprocating parts, *i.e.* reciprocating portion of the connecting rod, piston, rings, gudgeon, piston rod and crosshead (if any), at crankpin radius.

(iv) *Moment of Inertia by Experiment.*—The moment of inertia of a complicated member such as an armature or rotor may be found quite easily by experiment. Referring to Fig. 24, the mass is slung by two equal length wires from a beam, the axis of the mass being vertical, and is given a small initial twist and allowed to oscillate torsionally. The time taken to make one complete oscillation is then noted. If this time is t seconds

$$t = 2\pi\sqrt{\frac{lr^2}{gx^2}}$$

where l is the length of each suspension wire in feet.
x is the distance of each wire from the central axis.
g is 32·2.
r is the radius of gyration in feet.

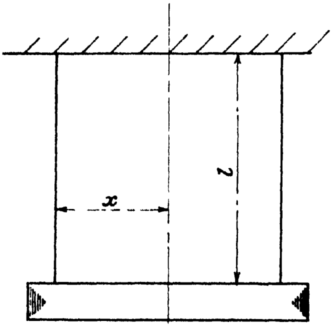

Fig. 24.—Moment of Inertia by Experiment.

Thus the radius of gyration is found and, the weight having previously been noted, the moment of inertia is found from the usual formula $I = \frac{Wr^2}{g}$.

THE DISTURBING FORCES—CRITICAL SPEEDS

Harmonic Analysis.—On page 10 it was shown that resonance conditions can be produced in the torsional pendulum (Fig. 1) by oscillating it at the correct frequency. In the diesel engine similar conditions can prevail, as the torsional elastic system, comprising the crankshaft and its associated revolving and reciprocating masses, is acted upon by the periodically varying cylinder torque.

This cylinder torque is made up of a large number of harmonics of varying amplitude and frequency. Each of these harmonics is theoretically capable of producing resonance at the appropriate number of revolutions per minute, the actual degree of resonance produced being dependent on the magnitude of the harmonic and the amount of damping present.

The turning effort curve may be resolved into its harmonic components by means of the Fourier analysis. Fourier's theorem states that any periodic function such as

$$y = \Phi(\theta)$$

which repeats itself in the intervals 0 to 2π, 2π to 4π, etc., may be represented by the series

$$\begin{aligned} y = \Phi(\theta) = a_0 + a_1 \sin\theta + a_2 \sin 2\theta + \ldots a_p \sin p\theta \\ + b_1 \cos\theta + b_2 \cos 2\theta + \ldots b_p \cos p\theta \quad (37) \end{aligned}$$

or alternatively by

$$\begin{aligned} y = \Phi(\theta) = a_0 + x_1 \sin(\theta + \varphi_1) \\ + x_2 \sin(2\theta + \varphi_2) + \ldots \quad (38) \end{aligned}$$

where

$$x_1 = \sqrt{a_1^2 + b_1^2}$$

$$\tan\varphi_1 = b_1/a_1$$

In these expressions, a_0 represents the mean height of the curve, whilst the remaining terms of the series are known as the harmonic components.

To perform the harmonic analysis, the mean height a_0 is first determined by finding the area by planimeter and then dividing by the length. To find the harmonics, the period is divided into $2h$ equal parts and ordinates erected.

Then the values of the harmonics are given by*

$$a_n = \frac{1}{h}\Sigma A_s \sin \frac{sn\pi}{h}$$

$$b_n = \frac{1}{h}\Sigma A_s \cos \frac{sn\pi}{h}$$

where A_s is the ordinate at $\theta = s\pi/h$.

It will be seen from the above that the number of ordinates must be twice the order of the highest harmonic to be extracted. Also, it can be proved that the larger the number of ordinates, the more nearly will the calculated value of the harmonic approach the true value.

A typical example of harmonic analysis is given in Fig. 25 in which the turning effort curve is that of a four cycle diesel engine. The fundamental harmonic is designated " Order ½ " since its frequency is one half engine speed, the second harmonic " Order 1," and so on.

When carrying out the harmonic analysis of a turning effort curve, the inertia forces may be included with those of the gas pressure and the resultant diagram analysed, or the two curves may be analysed separately and the appropriate harmonics added vectorially. In the example given, inertia has been included.

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* For a very complete treatise on Harmonic Analysis see "*Fourier's Theorem and Harmonic Analysis*," by Eagle.

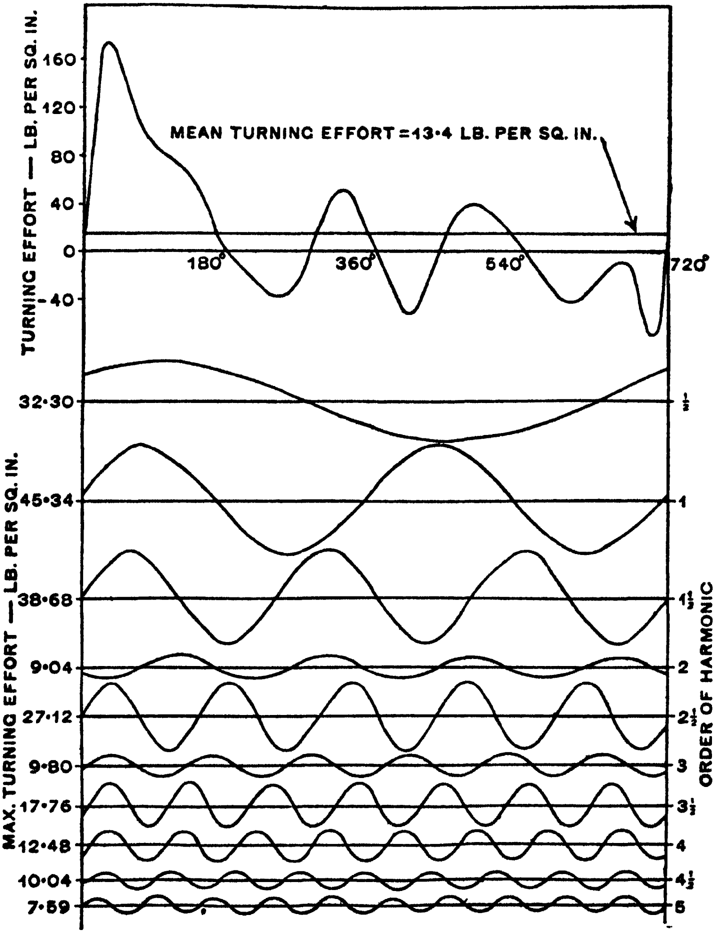

Fig. 25.—Harmonic Analysis of Turning Effort Diagram.

To evaluate the inertia harmonics separately, it is not necessary to analyse the inertia curve graphically as in the case of the gas pressure. All that is necessary is to set down the expression for the turning effort due to inertia, and expand it in a series. This series will be found to take the form

$$y_1 = \frac{W_c \omega^2 r}{g}(c_1 \sin \theta + c_2 \sin 2\theta + c_3 \sin 3\theta + \ldots c_p \sin p\theta) \quad . \quad (39)$$

i.e. there are no cosine components and no half order components.

In the above expression,

W_c = weight of reciprocating parts in lbs.
$g = 32{\cdot}2$
r = crank radius in feet.
ω = angular velocity of crankshaft in radians per second.

Tables IV and V give the harmonic components for gas pressure and inertia respectively when the curves are analysed separately. The gas pressure harmonics are for a mean indicated pressure of 90 lbs. per sq. in. Both sets of harmonics are for a crank/connecting rod ratio of 1/4·5.

The harmonic torques due to gas pressure then are :—

Sine Component $= H_s = a_n \times \pi d^2/4 \times r$ ft. lbs.
Cosine Component $= H_c = b_n \times \pi d^2/4 \times r$ ft. lbs.
Resultant Component $= H_r = x_n \times \pi d^2/4 \times r$ ft. lbs.

where d is the cylinder diameter in inches and r is the crank radius in feet.

TABLE IV

Harmonic Order n	Sine Term a_n	Cosine Term b_n	Resultant Term x_n
½	22·2	−23	32
1	32·5	−13	35
1½	35	− 2·8	35·1
2	32·4	6	33
2½	28	6·4	28·7
3	23·5	5·8	24·2
3½	19·4	5·4	20
4	13·6	4·8	14·4
4½	10·4	3·85	11·1
5	7·7	3·3	8·4
5½	5·55	2·8	6·2
6	3·69	2·57	4·5
6½	2·7	2·2	3·49
7	2·02	1·94	2·8
7½	1·57	1·58	2·23
8	1·25	1·27	1·78
8½	1·0	1·05	1·45
9	·82	·87	1·20
9½	·70	·70	·99
10	·57	·57	·81
10½	·52	·44	·68
11	·46	·36	·58
11½	·40	·27	·48
12	·37	·20	·42

The harmonic torque due to inertia is :—

$$\text{Sine Component} = H_i = c_n \times W_r\omega^2/g \times r^2 \text{ ft. lbs.}$$

The resultant harmonic torque due to the combination of the gas pressure and inertia harmonics is given by

$$H_m = \sqrt{(H_s + H_i)^2 + H_c^2} \text{ ft. lbs.} \quad . \quad . \quad (40)$$

TABLE V

Harmonic Order n	Sine Term c_n
1	+ ·056
2	— ·500
3	— ·170
4	— ·013
5	+ ·002
6	+ ·0001
7	— ·00002
8	0

Critical Speeds.—When the crankshaft is revolving at such a speed that one of these harmonics synchronises with the natural frequency, resonance occurs, and it is said to be at a critical speed of that particular order. Thus in a four cycle engine it is possible to have critical speeds of every integral order and half order and in two cycle engines, by similar reasoning, critical speeds of every integral order, the order number being determined by the number of vibrations occurring per revolution of the crankshaft.

Now it will at once be apparent that the magnitude of the vibration in a multi-cylinder engine differs widely for the various critical speeds, even when allowance is made for the difference in value of the harmonic forcing torques. This is due to the fact that, for some orders, the harmonics assist one another in producing vibration, whilst for others, the reverse process occurs. It therefore becomes necessary, not only to classify the critical speeds, but to obtain some idea of their relative capacity for producing vibration.

Let an nth order harmonic torque H_n be applied at the rth cylinder where the ordinate on the normal elastic curve has the value a_r.

The work done, *i.e.* the kinetic energy, at this point is proportional to

$$H_n a_r$$

Then for the entire group of cylinders, assuming the torque H_n to be applied at each cylinder centre line, the total kinetic energy is given by

$$E = \text{a constant} \times H_n \Sigma a_r \quad . \quad . \quad (41)$$

where Σa_r represents the vector sum of the normal elastic curve ordinates at which the torque is applied.

We thus reach the important conclusion that the magnitude of the vibration in a critical speed is directly dependent on the value of $H_n \Sigma a_r$, that is, on the harmonic forcing torque, on the shape of the normal elastic curve, and on the crank sequence and firing order.

The value of Σa is most easily evaluated by means of phase and vector diagrams, a typical example of which is shown in Fig. 26. The system is that of a six cylinder four stroke engine with a firing order of 1, 5, 3, 6, 2, 4. The normal elastic curve is as shown.

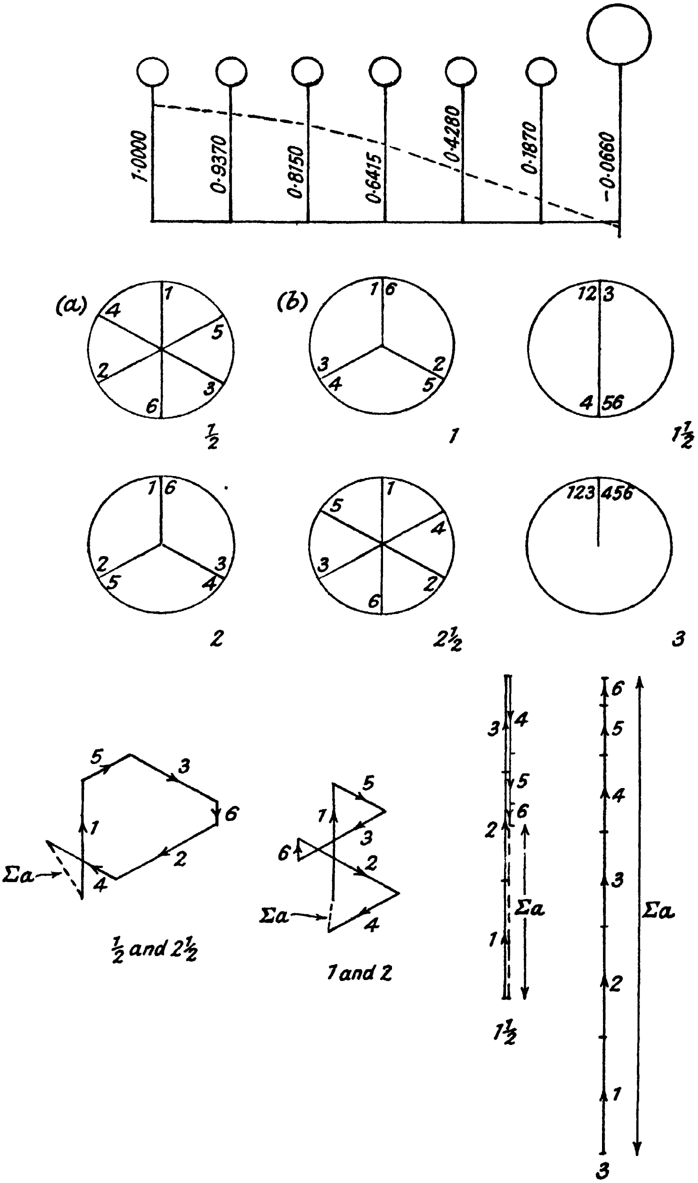

Fig. 26.—Phase and Vector Diagrams, Normal Six Cylinder Engine.

The explanation of the diagrams is as follows. For each order of critical speed, a phase diagram is constructed, 360° representing one complete vibration. For the half order critical speed, the cylinders come into action in the phase diagram at equal intervals and in their correct firing order as shown at (*a*). For the first order critical speed, since one vibration is completed in half the time, the angles will be doubled as shown at (*b*). For orders 1½, 2, etc., the angles for order ½ are multiplied by 3, 4, etc., up to and including order 3. After order 3, the diagrams repeat, *i.e.* orders ½, 3½, 6½, etc., orders 1, 4, 7, etc., have the same phase diagrams respectively.

The vector diagrams are shown beneath the phase diagrams in Fig. 26. The vectors are drawn parallel to their respective cylinders in the phase diagrams and of lengths equal to their respective ordinates in the normal elastic curve. The closure of each vector diagram gives the value of Σa, for that particular order or orders.

It will be noticed that the phase diagrams for orders ½ and 2½ and for orders 1 and 2, respectively, are mirror images. Only one vector diagram is therefore needed for each pair.

It will now be seen that for critical speeds of orders 3, 6, 9, 12, etc., the harmonic torques are all in phase and tend to oscillate the shaft in its normal mode of vibration. These critical speeds are the most powerful and are known as "major" critical speeds, all others being termed "minors." For the type of system under consideration, which incidentally is the type most frequently encountered in diesel engineering practice, major critical speeds may be defined as those whose order numbers are equal to or multiples of the number of power strokes per revolution.

It should be clearly understood that the above definition of major critical speeds does not apply to all systems. Whether a critical speed is major or minor depends entirely on the shape of the normal elastic curve. This will be made clear by reference to Fig. 27, which shows the normal elastic curve, phase and vector diagrams for a system which differs from that of Fig. 26 only in that a flywheel has been placed either end. The node, in this case, lies in the centre of the engine crankshaft and the vector diagrams show that for such conditions, orders $1\frac{1}{2}$, $4\frac{1}{2}$, $7\frac{1}{2}$, etc., are the major critical speeds, all others being minors.

The Effect of Firing Order.—In the previous section it was shown that the firing order determines the sequence of the vectors in the phase and vector diagrams. Hence the effect of the harmonic forcing torques, as represented by the closure of the vector diagram, can be reduced or increased for certain critical speeds by a suitable modification to the crank sequence and firing order.

This method of modifying the magnitude of critical speeds is, however, very limited in its application. Crank sequence can very rarely be altered without seriously upsetting engine balance and only in certain cases can the firing order be altered without altering the crank sequence. The only cases where the method is extensively used are those of six and eight cylinder four stroke engines, in which several firing orders are available for the same crank sequence.

The effect of altering the firing order is best demonstrated by an actual example. The equivalent

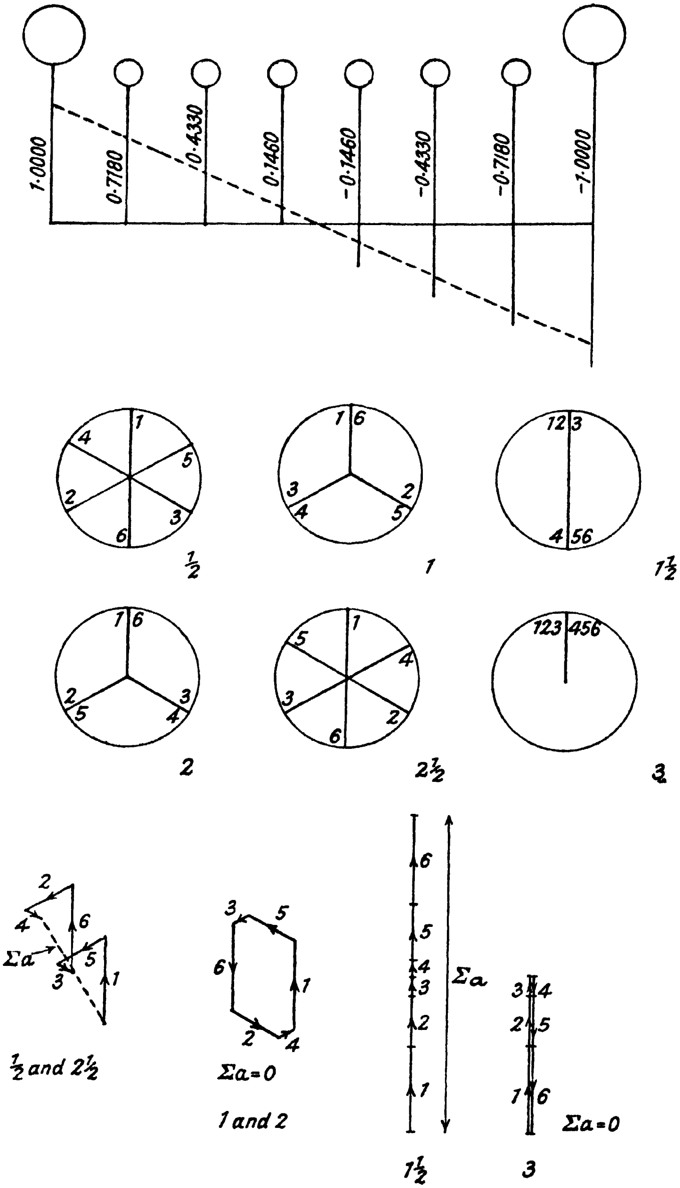

Fig. 27.—Phase and Vector Diagrams. Flywheel Each End.

dynamical system is shown in Fig. 28, together with the normal elastic curve and phase and vector diagrams for firing orders 1, 5, 3, 6, 2, 4, and 1, 2, 4, 6, 5, 3, respectively. The engine, which has a one node natural frequency of 9,050 v.p.m., is normally expected to run at 1,200 r.p.m.

For the above natural frequency, the critical speeds in the neighbourhood of the running speed (obtained by dividing the natural frequency by the order number) are :—

Critical Speed	*Order*
1510 r.p.m.	6
1392 ,,	6½
1290 ,,	7
1205 ,,	7½
1130 ,,	8
1065 ,,	8½
1005 ,,	9

We thus have 6th and 9th order major criticals at 1510 and 1005 r.p.m. respectively, whilst there is a 7½th order minor critical practically on the running speed.

It has previously been shown that the magnitude of the vibration in a critical speed is proportional to

$$H_n \Sigma a_r$$

where H_n is the harmonic forcing torque and Σa_r is the closure of the vector diagram. The system of Fig. 28 has therefore been analysed on this basis, values of $H_n \Sigma a_r$ having been found for each particular order.

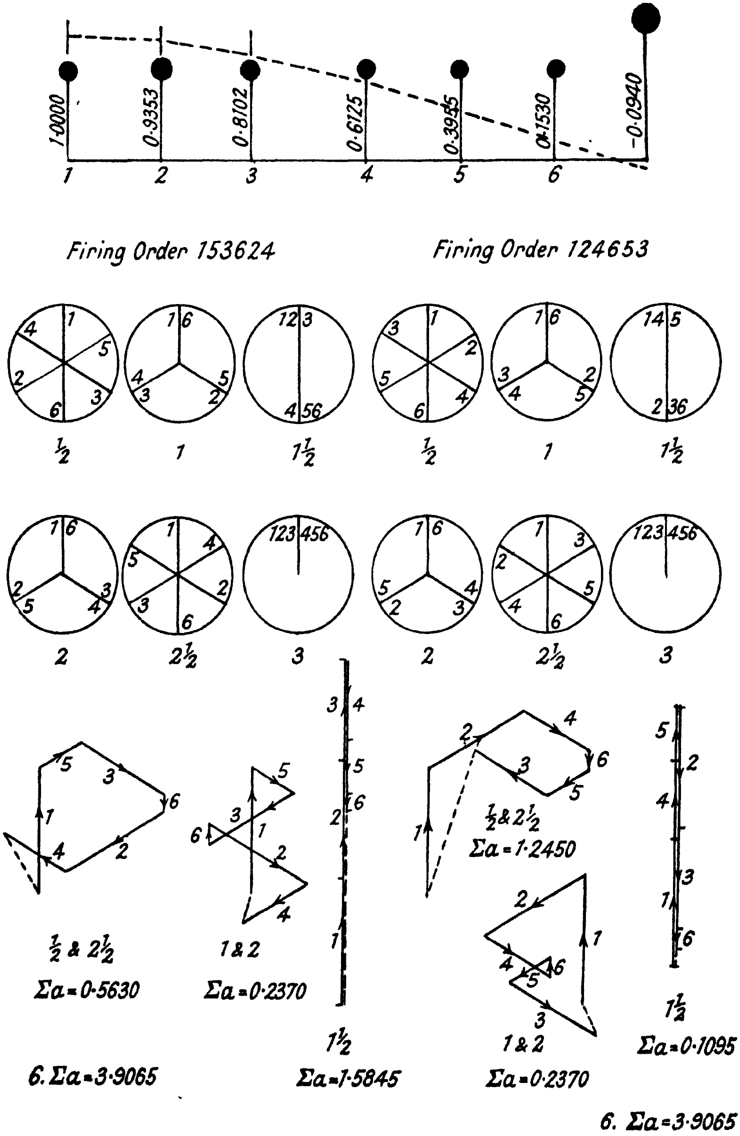

Fig. 28.—Phase and Vector Diagrams for Different Firing Orders.

The work is best arranged in tabular form as shown in Table VI. This table gives the values of $H_n \Sigma a_r$ for the two firing orders in question.

Thus, from the fifth column, for a firing order 1, 5, 3, 6, 2, 4, the 7½th order minor critical, which lies on the running speed, is almost as important as the 9th order major. Turning now to the seventh column, by altering the firing order to 1, 2, 4, 6, 5, 3, the 7½th order critical almost disappears and the 6½th and 8½th become of importance. For a running speed of 1200 r.p.m., therefore, the firing order 1, 2, 4, 6, 5, 3 is preferable.

TABLE VI

Critical Speed	Order	H_n	Firing Order 1, 5, 3, 6, 2, 4		Firing Order 1, 2, 4, 6, 5, 3	
			Σa_r	$H_n \Sigma a_r$	Σa_r	$H_n \Sigma a_r$
1510 r.p.m. . .	6	4·5	3·9065	17·5	3·9065	17·5
1392 ,, . .	6½	3·49	·5630	1·96	1·2450	4·34
1290 ,, . .	7	2·80	·2370	·663	·2370	·663
1205 ,, . .	7½	2·23	1·5845	3·54	·1095	·245
1130 ,, . .	8	1·78	·2370	·422	·2370	·422
1065 ,, . .	8½	1·45	·5630	·816	1·2450	1·80
1005 ,, . .	9	1·20	3·9065	4·7	3·9065	4·7

As would be expected from the phase and vector diagrams, the major critical speeds are unaffected by firing order.

VIBRATION AMPLITUDES IN CRITICAL SPEEDS

Energy Balance.—On page 5 it was assumed, for mathematical convenience, that damping was proportional to the velocity of the vibrations. By this means the equation of motion was made linear and expressions were obtained for the equilibrium amplitude and the dynamic magnifier and the resonance curves of Fig. 3 were constructed.

In the case of a multi-mass system, of which the diesel engine is a particular example, the inclusion in the equations of motion of a damping factor, even in its simplest linear form, leads to appalling mathematical complications, and the labour involved in obtaining a solution is not justified by the accuracy of the results obtained. Fortunately, however, the condition of resonance permits a different treatment of the problem and allows non-linear damping conditions to be taken into account without much difficulty.

The principle of the method used is that during steady conditions of resonance, the energy input per cycle due to the forcing torques must be equal to that absorbed by damping.

Damping may arise from a number of sources and may be either internal or external, that is, in the engine itself or in the driven machinery. In the case of external damping, the most usual sources are hysteresis losses in the driven shafting, and losses due to torque variation in a driven member such as a marine propeller, electric generator, etc. In the case of internal damping, the losses are divided between hysteresis and oil film dissipation at the main bearings, the latter probably being the more important.

Work Done per Vibration at the Cylinder Centre Line.—Let a harmonic torque of the form

$$T = H \sin \omega t$$

be applied at the cylinder centre line. Then the motion produced will be of the form

$$\alpha = a \sin (\omega t - \varepsilon)$$

Fig. 29 shows the torque and the motion plotted vectorially, a lagging behind H by the angle ε.

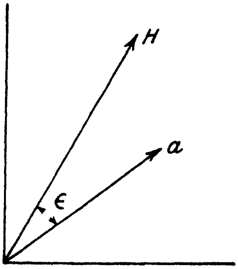

Fig. 29.—Vector Diagram of Torque and Motion.

Let the work done per vibration be δw. Then

$$\begin{aligned}\delta w &= \int_0^{2\pi} T d\alpha \\ &= \int_0^{2\pi/\omega} Ha\omega \sin \omega t \cos (\omega t - \varepsilon) dt \\ &= \pi Ha \sin \varepsilon \quad . \quad . \quad . \quad . \quad . \quad (42)\end{aligned}$$

During conditions of resonance, $\varepsilon = \pi/2$ and therefore $\sin \varepsilon = 1$ so that

$$\delta w = \pi Ha \quad . \quad . \quad . \quad (43)$$

Work Absorbed by Damping at the Cylinder Centre Line (*Damping Proportional to Velocity*).– Again let a be the maximum amplitude at the cylinder centre line.

Let the damping torque be $\mu\frac{d\alpha}{dt}$, where μ is the coefficient of friction, and $\alpha = a \sin \omega t$.

Then for the work absorbed per vibration

$$\begin{aligned} \delta w_1 &= \int_0^{2\pi} \mu \frac{d\alpha}{dt}\, d\alpha \\ &= \int_0^{2\pi/\omega} \mu a^2\omega^2 \cos^2 \omega t\, dt \\ &= \mu\pi a^2\omega \quad . \quad . \quad . \quad . \quad . \quad (44) \end{aligned}$$

Extension to Multi-Crank Engine.—Coming now to a multi-crank engine, the total work done at resonance by the nth order harmonic torque will be, as shown on page 58,

$$\delta w = \pi H_n \Sigma a \quad . \quad . \quad . \quad (45)$$

or, if θ be the measured amplitude at No. 1 cylinder centre line, where the ordinate of the normal elastic curve is unity

$$\delta w = \pi H_n \theta \Sigma a \quad . \quad . \quad . \quad (46)$$

Similarly, the work absorbed by damping, assumed to be equally distributed between the cylinders, is given by

$$\delta w_1 = \pi \mu \omega \theta^2 \Sigma a^2 \quad . \quad . \quad . \quad (47)$$

Now at resonance δw equals δw_1, therefore

$$\pi H_n \theta \Sigma a = \pi \mu \omega \theta^2 \Sigma a^2$$

whence

$$\mu = \frac{H_n \Sigma a}{\omega \theta \Sigma a^2} \quad . \quad . \quad . \quad (48)$$

This expression enables the damping factor for a given engine to be calculated in terms of the measured amplitude at No. 1 cylinder and *vice versa.*

Referring back to page 9, it was shown that for resonance conditions in the torsional pendulum, the dynamic magnifier is $1/K$, which equals $I\omega/\mu$. This same conception can be usefully applied to the multi-crank engine having equal moments of inertia per line, as follows.

Dividing both sides of Equation (48) by $I\omega$ and inverting, we have

$$\text{Dynamic Magnifier} = \frac{I\omega}{\mu} = \frac{I\omega^2\theta\Sigma a^2}{H_n\Sigma a} \quad . \quad . \quad (49)$$

Damping Within the Engine—Shannon's Method.—In a paper published in the *Journal of the Royal Technical College, Glasgow,* 1933, and later read in an amplified form before the Institution of Mechanical Engineers,* Mr. J. F. Shannon published the results of his researches on torsional vibration, and he came to the conclusion that the vibrational energy at resonance was dissipated mainly by the induced actions on the oil films at the big end and main bearings. Losses due to hysteresis were found to be such a small percentage of the total losses that they could be neglected. Therefore by assuming that the overall damping was viscous in character, the dynamic magnifier† was correlated to the vibration form of the engine and a relationship was obtained whereby the amplitude in a critical speed could be reasonably determined.

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* "Engine Damping Factors in Torsional Oscillation and the Effect of Vibration Form," by J. F. Shannon, *Jl.Roy.Tech.Coll.,Glasgow,* January, 1933.

† Called by Shannon "the Engine Damping Coefficient."

It was found that for different engine systems, there was a wide variation in the value of the dynamic magnifier. The latter was, however, found to be dependent on the shape of the normal elastic curve, so that by plotting $I\omega/\mu$ against $\Sigma a/n$, where n is the number of cylinders, a reasonable classification of a number of test results was obtained. From these results, the curve shown in Fig. 30 was put forward as a guide in design.

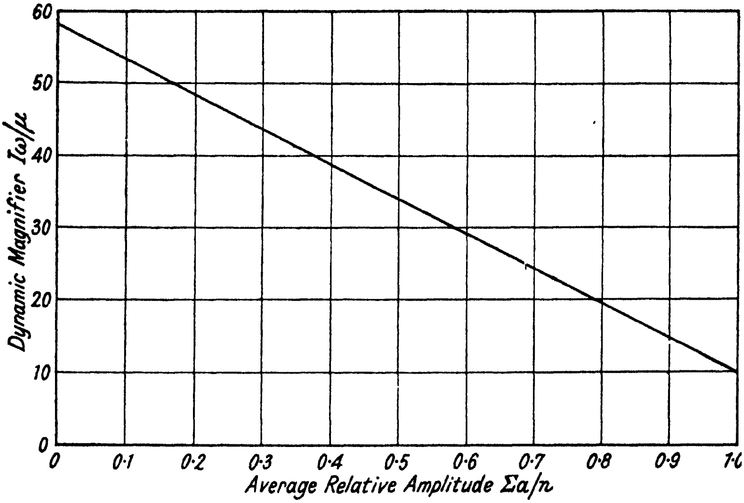

Fig. 30.—Shannon's Line.

The method of using the curve is best shown by the following worked example of a six cylinder diesel engine. The specification is as follows :—

Moment of inertia per cylinder line . . . . . = 1·2196 lbs. ft.^{2} sec.^{2}
Moment of inertia of flywheel = 22·6 lbs. ft.^{2} sec.^{2}

Equivalent length between cylinder centres . . = 1·58 ft. of 4¾" dia. shaft.

Equivalent length between No. 6 cylinder and flywheel = 1·657 ft. of 4¾" dia. shaft.

The equivalent system as set out for checking is as shown in Fig. 31.

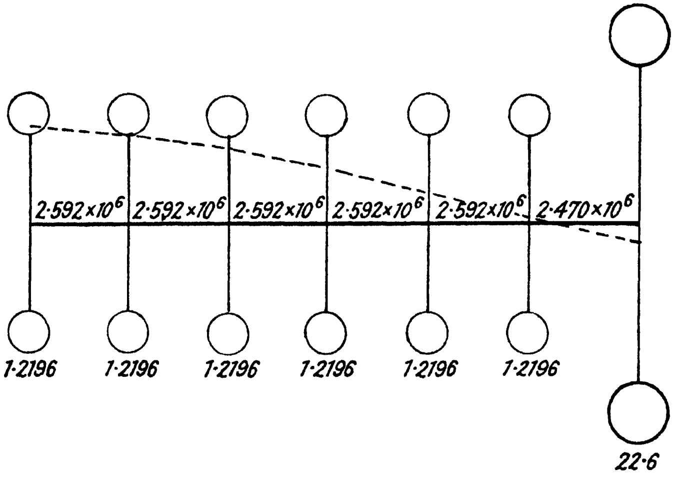

Fig. 31.—Six Cylinder Diesel Engine.

Using the Successive Approximation method of calculation, the table is as follows (see page 72) :—

The natural frequency

$$f = \frac{393}{2\pi} = 62{\cdot}6 \text{ v.p.s.}$$

and the normal elastic curve is as shown in Fig. 31.

TABLE VII

		$\omega = 393$			$\omega^2 = 154500$			
Mass No.	I	$I\omega^2$	a	$I\omega^2 a$	$\Sigma I\omega^2 a$	R	$\frac{\Sigma I\omega^2 a}{R}$	Stress tons sq. in.
1.	1·2196	$1·885 \times 10^5$	1·0000	$1·885 \times 10^5$	$1·885 \times 10^5$	$2·59 \times 10^6$	·0728	·842
2.	1·2196	$1·885 \times 10^5$	·9272	$1·747 \times 10^5$	$3·632 \times 10^5$	$2·59 \times 10^6$	·1402	1·62
3.	1·2196	$1·885 \times 10^5$	·7870	$1·482 \times 10^5$	$5·114 \times 10^5$	$2·59 \times 10^6$	·1975	2·29
4.	1·2196	$1·885 \times 10^5$	·5895	$1·110 \times 10^5$	$6·224 \times 10^5$	$2·59 \times 10^6$	·2403	2·78
5.	1·2196	$1·885 \times 10^5$	·3492	$·658 \times 10^5$	$6·882 \times 10^5$	$2·59 \times 10^6$	·2657	3·07
6.	1·2196	$1·885 \times 10^5$	·0835	$·157 \times 10^5$	$7·039 \times 10^5$	$2·47 \times 10^6$	·2850	3·15
7.	22·6	$35·00 \times 10^5$	—·2015	$-7·052 \times 10^5$	$-·013 \times 10^5$	...	...	...

It is now required to find the amplitude in torsional vibration due to the sixth order critical speed, which occurs at

$$\frac{62\cdot6 \times 60}{6} = 626 \text{ r.p.m.}$$

The "form factor"

$$\Sigma a/n = \frac{3\cdot7364}{6} = 0\cdot6227$$

From Fig. 30, the dynamic magnifier $I\omega/\mu$ corresponding to this form factor is 28.

Hence

$$\frac{I\omega}{\mu} = 28 = \frac{I\omega^2\theta\Sigma a^2}{H_6\Sigma a}$$

so that

$$\theta \quad \frac{28H_6\Sigma a}{I\omega^2\Sigma a^2}$$

The engine in question has a bore of 6$\frac{3}{8}$ ins. and a stroke of 9 ins., so that

$$H_6 = \frac{4\cdot50 \times 31\cdot92 \times 4\cdot5}{12} = 54 \text{ ft. lbs.}$$

$$\theta = \frac{28 \times 54 \times 3\cdot7364}{1\cdot2196 \times 154500 \times 2\cdot9555}$$

$$= 0\cdot0102 \text{ radian} = 0\cdot584 \text{ degree.}$$

From the last column of Table VII the corresponding stress at the node will be 0·584 × 3·15 = 1·84 tons per sq. in.

The actual measured amplitude at No. 1 cylinder was 0·437°, the corresponding stress at the node being 1·37 tons per sq. in.

Hysteresis Damping.—The shafting material of a diesel engine and its associated drives, when subjected to repeated cycles of stress as in torsional vibration, does not obey Hooke's law exactly. The stress-strain relationship of steel under these conditions is generally

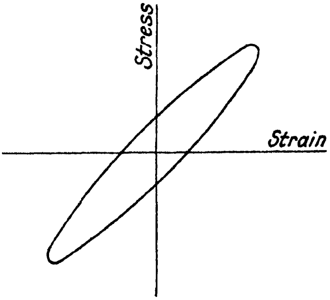

Fig. 32.—Hysteresis Loop.

as shown in Fig. 32, the area of which gives the amount of energy dissipated per cycle per cubic inch of material.

When the stress is low, the area of this " hysteresis loop " is so small as to be negligible. As the stress range is increased, however, the area increases enormously, so that the curve of energy dissipated, plotted against stress range, is as shown in Fig. 33.

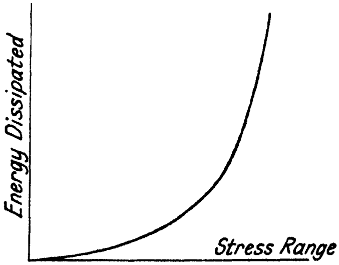

Fig. 33.—Typical " Energy-Stress " Curve.

A number of authorities have investigated the question of hysteresis loss in crankshaft steels. The most recent and probably the most comprehensive series of experiments were conducted by Dr. Dorey, who presented his results in a paper to the Institution of Mechanical Engineers in November, 1932.* Dr. Dorey found that all the steels tested showed a quasi-critical point below which the rate of dissipation of energy varied approximately as the third power of the stress range and above which the rate varied approximately as the tenth power of the stress range for primary loading and as the sixth power of the stress range for secondary and subsequent loadings.

The following expression for hysteresis energy per unit volume per cycle was evolved :—

$$\delta w = Z\left(\frac{S}{S_c}\right)^{\gamma} \quad . \quad . \quad . \quad (50)$$

where δw = hysteresis energy per unit volume per cycle (inch lbs. per cubic inch).
Z = hysteresis constant.
S = half the stress range for alternating stress.
S_c = half the critical stress range.
γ = hysteresis exponent.

For a uniform hollow circular shaft, of external and internal radii r_1 in. and r_2 in., respectively, and length l in., the hysteresis loss per cycle is

$$\delta w = \frac{2\pi}{(2+\gamma)} \cdot Z \cdot \frac{r_1^{2+\gamma} - r_2^{2+\gamma}}{r_1^{\gamma}}\left(\frac{S}{S_c}\right)^{\gamma} \cdot l$$

which becomes

$$\delta w = \frac{2\pi}{(2+\gamma)} \cdot Z \cdot r^2\left(\frac{S}{S_c}\right)^{\gamma} \cdot l \quad . \quad . \quad (51)$$

for a solid shaft of radius r.

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* "Elastic Hysteresis in Crankshaft Steels," Dorey, *Proc.I.Mech.E.*, 1932, Vol. 123, p. 479.

For a normal crankshaft material of 0·21 per cent. carbon steel having a maximum tensile stress of 30 tons per sq. inch,

(*a*) For primary loading

Z = 0·18.
S_c = 6 tons per sq. inch.
γ = 2·8 for values of S below S_c.
10·0 for values of S above S_c.

(*b*) For secondary and subsequent loadings above the critical stress range

Z = 0·74
S_c = 6 tons per sq. inch.
γ = 3·2 for values of S below S_c.
6·6 for values of S above S_c.

As the endurance limit for crankshaft steel subjected to repeated alternating stress is in the neighbourhood of the value of S_c, category (*a*) above may be said to represent hysteresis conditions for torsional vibration problems in diesel engines.

Thus for a hollow circular shaft

$$\delta w = \frac{0{\cdot}18\pi}{2{\cdot}4} \cdot \frac{r_1^{4\cdot8} - r_2^{4\cdot8}}{r_1^{2\cdot8}} \cdot \left(\frac{S}{6}\right)^{2\cdot8} . \, l \quad . \quad (52)$$

and for a solid circular shaft

$$\delta w = \frac{0{\cdot}18\pi}{2{\cdot}4} \, . \, r^2 \, . \left(\frac{S}{6}\right)^{2\cdot8} . \, l \text{ inch lbs.} \quad . \quad (53)$$

Ker Wilson's Method.—In an article in *Engineering* of February 16, 1934, W. Ker Wilson obtained a simplified expression, based on Dorey's researches, which gives the stress in a critical speed when hysteresis is assumed to be the principal damping influence. The expression is as follows :—

$$S = 602\ \sqrt[18]{S_0}\ \text{lbs. per sq. in.} \qquad . \quad (54)$$

for 30 ton plain carbon steel not previously subjected to cycles of stress above the critical range and for values of S below S_c, where

S is the maximum vibration stress.

S_0 is the " equilibrium " stress.

The value of S_0 is obtained from a formula for the " equilibrium amplitude " a_0, which, converted to the symbols used in this book and applied to the type of engine under consideration, is

$$a_0 = \frac{57{\cdot}3 H_n \Sigma a}{\omega^2 \Sigma (I a^2)}\ \text{degrees} \qquad . \quad . \quad . \quad (55)$$

where $\Sigma a = a_1 + a_2 + \ldots\ldots a_6$

$$I a^2 \Sigma = I_1 a_1^2 + I_2 a_2^2 + \ldots\ldots I_7 a_7^2.$$

Again using the example of Fig. 31,

$$a_0 = \frac{57{\cdot}3 \times 54 \times 3{\cdot}7364}{154500 \times 4{\cdot}52} = 0{\cdot}0165^\circ$$

Whence the equilibrium stress

$$S_0 = 0{\cdot}0165 \times 3{\cdot}15 = 0{\cdot}052 \text{ tons per sq. in.}$$
$$= 116{\cdot}5 \text{ lbs. per sq. in.}$$

The maximum vibration stress

$$S = 602\ \sqrt[18]{116{\cdot}5} = 8464 \text{ lbs. per sq. in.}$$
$$= 3{\cdot}78 \text{ tons per sq. in.}$$

Summary

	Actual Measurement.	Shannon's Method.	Ker Wilson (Hysteresis).
Stress (tons/sq. in.) .	1·37	1·84	3·78
Correction Factor .	..	0·75	0·363

General Comments.—From the results summarised above, the following conclusions may be drawn :—

(*a*) The "Shannon" method of calculation gives a reasonably close approximation to the actual measured stress.

(*b*) Hysteresis accounts for about one-third of the total damping present.

It must not be assumed from the conclusions reached above that hysteresis plays no part in engine damping, but numerous investigators* have found that it is necessary to multiply hysteresis losses by a factor ranging between 2 and 3 to make their calculations agree with the results obtained from torsiograph records. The Shannon method, whilst apparently ignoring hysteresis, actually takes it into account in the general over-all damping factor, the greater portion of which is found to be due to the viscous losses already mentioned. The author's experience is that the above conclusions are applicable to the majority of diesel engines.

External Damping—Hysteresis.—The energy absorbed by hysteresis in the driven shafting is calculated by means of Equation (52) or (53) by inserting the appropriate value of S from Column 9, Table VII. This gives the energy loss per degree twist at No. 1

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* See discussion on Dorey's Paper, *loc. cit.*

crank. If the shaft is of variable section, care must be taken to assess the stress and consequent loss in each individual section.

Damping Effect of a Marine Propeller.—J. F. Shannon* quotes the following formula for calculating the damping factor of a marine propeller

$$\mu = \frac{34{\cdot}4T}{N} \quad . \quad . \quad . \quad (56)$$

for normal propellers and slips where

T = propeller torque in ft. lbs.
N = r.p.m.
μ = damping factor in ft. lb. sec. units.

Damping Effect of an Electric Generator.—Lewis† gives the following approximate formula for calculating the damping factor of an electric generator.

$$\mu = \frac{422{\cdot}7E^2I_a}{2\pi(E - E_t)N^2} \quad . \quad . \quad (57)$$

where E = total E.M.F. of generator (terminal voltage plus internal drop).
E_t = back E.M.F. of driven machinery (not an external resistance).
I_a = current.
N = r.p.m.
μ = damping factor in ft. lb. sec. units.

The method of incorporating these various damping factors is again best shown by means of a worked example. Stated briefly the method consists of finding the energy dissipated in the engine and in the driven unit each in terms of the amplitude θ at the forward end of the engine. The sum of these is equated to the input energy from the harmonic forces—also in terms of θ—and the resulting equation is solved for θ. A complete example of a diesel electric set is worked out in the following section.

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* *Loc. cit.*

† Lewis, *Trans. Amer. Soc. Naval Arch. and Mar. Eng.*, New York, 1925.

EXAMPLE—*DIESEL-ELECTRIC SET*

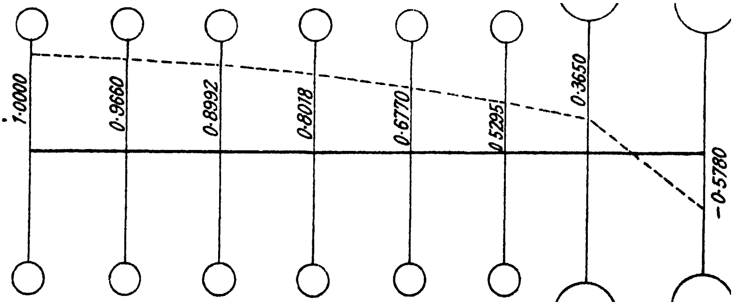

Fig. 34.—Diesel-Electric Set with Normal Elastic Curve.

The equivalent system as set out for checking is as shown in Fig. 34. The specification is as follows :—

Bore and Stroke = 11 in. × 14½ in.
Moment of inertia per cylinder line . . . . . = 6·63 lbs. ft.^{2} sec.^{2}
Moment of inertia of flywheel = 376 lbs. ft.^{2} sec.^{2}
Moment of inertia of generator = 298 lbs. ft.^{2} sec.^{2}
Stiffness between cylinder centres . . . . . = 14·1 × 10^6 ft. lbs. per radian.

Stiffness between No. 6 cylinder and flywheel . = $14 \cdot 25 \times 10^6$ ft. lbs. per radian.

Stiffness of generator shaft . = $13 \cdot 0 \times 10^6$ ft. lbs. per radian.

Using the Successive Approximation method of calculation, the table is as follows (page 82) :—

$$\text{Natural frequency} = \frac{269}{2\pi} = 43 \text{ v.p.s.}$$

There is thus a 6th order critical at

$$\frac{43 \times 60}{6} = 430 \text{ r.p.m.}$$

The stresses in the final column are calculated for an 8 in. journal shaft.

The stress per degree in the $7\frac{3}{8}$ in. dia. generator shaft will be

$$11 \cdot 5 \times \left(\frac{8}{7\frac{3}{8}}\right)^3 = 14 \cdot 6 \text{ tons per sq. in.}$$

It is required to find the amplitude due to this 6th order critical speed.

Let θ radians be the amplitude at No. 1 crank. Then, from Equation (46) :—

The input work per vibration

$$= \pi H_6 \theta \Sigma a$$

$$= \pi \times 4 \cdot 5 \times 95 \cdot 033 \times 7 \cdot 25 \times 4 \cdot 8735\theta$$

$$= 475000\theta \text{ inch lbs.}$$

Damping is divided between three sources, namely,

(*a*) Overall damping in the engine.
(*b*) Hysteresis damping in the generator shaft.
(*c*) Damping in the generator.

TABLE VIII

$\omega = 269$				$\omega^2 = 72361$				
Mass No.	I	$I\omega^2$	a	$I\omega^2 a$	$\Sigma I\omega^2 a$	R	$\frac{\Sigma I\omega^2 a}{R}$	Stress tons sq. in.
1.	6·63	$\cdot 48 \times 10^6$	1·0000	$\cdot 48 \times 10^6$	$\cdot 48 \times 10^6$	$14\cdot 1 \times 10^6$	·0340	·45
2.	6·63	$\cdot 48 \times 10^6$	·9660	$\cdot 463 \times 10^6$	$\cdot 943 \times 10^6$	$14\cdot 1 \times 10^6$	·0668	·88
3.	6·63	$\cdot 48 \times 10^6$	·8992	$\cdot 432 \times 10^6$	$1\cdot 375 \times 10^6$	$14\cdot 1 \times 10^6$	0974	1·285
4.	6·63	$\cdot 48 \times 10^6$	·8018	$\cdot 385 \times 10^6$	$1\cdot 760 \times 10^6$	$14\cdot 1 \times 10^6$	1248	1·650
5.	6·63	$\cdot 48 \times 10^6$	·6770	$\cdot 325 \times 10^6$	$2\cdot 085 \times 10^6$	$14\cdot 1 \times 10^6$	1475	1·95
6.	6·63	$\cdot 48 \times 10^6$	·5295	$\cdot 254 \times 10^6$	$2\cdot 339 \times 10^6$	$14\cdot 25 \times 10^6$	·1645	2·20
7.	376	$27\cdot 2 \times 10^6$	·3650	$9\cdot 93 \times 10^6$	$12\cdot 269 \times 10^6$	$13\cdot 0 \times 10^6$	9430	11·50
8.	298	$21\cdot 5 \times 10^6$	$-\cdot 5780$	$-12\cdot 43 \times 10^6$	$-\cdot 161 \times 10^6$	...	...	...

These dampings will be dealt with in turn.

(*a*) *Engine Damping* (*Shannon Method*).

$$\Sigma a = 4{\cdot}8735. \qquad \Sigma a^2 = 4{\cdot}1225.$$

$$\frac{\Sigma a}{6} = 0{\cdot}8122.$$

Whence from Fig. 30, $I\omega/\mu = 19{\cdot}2$.

$$\text{Therefore } \mu = \frac{6{\cdot}63 \times 269}{19{\cdot}2} = 92{\cdot}8 \text{ ft. lbs. per radian per sec.}$$

$$= 1113 \text{ inch lbs. per radian per sec.}$$

From Equation (47), the work absorbed by damping in the engine

$$= \pi\mu\omega\theta^2\Sigma a^2$$
$$= \pi \times 1113 \times 269 \times 4{\cdot}1225\theta^2$$
$$= 3{,}880{,}000\theta^2 \text{ inch lbs.}$$

(*b*) *Hysteresis Damping in the Generator Shaft.*—This shaft is $7\frac{3}{8}$ ins. uniform diameter and is 17 ins. long. The stress per degree twist at No. 1 cylinder is 14·6 tons per sq. in., whence the stress for θ radians at No. 1 cylinder is

$$14{\cdot}6 \times 57{\cdot}3 \times \theta = 837\theta \text{ tons per sq. in.}$$

Then from Equation (53)

$$\text{Hysteresis loss} = \frac{0{\cdot}18\pi}{2{\cdot}4} \,.\, r^2 \left(\frac{S}{S_e}\right)^{2\cdot8} .\, l$$
$$= \frac{0{\cdot}18\pi}{2{\cdot}4} \times 13{\cdot}6 \left(\frac{837\theta}{6}\right)^{2\cdot8} \times 17$$
$$= 54{,}500{,}000\theta^{2\cdot8} \text{ inch lbs.}$$

(*c*) *Generator Damping.*

$$\mu = \frac{422{\cdot}7E^2I_a}{2\pi(E - E_t)N^2}$$

The set was loaded by means of resistance plates, hence $E_t = 0$.

The load conditions were :—

$$E = 440 \text{ volts}, \; I_a = 500 \text{ amps.}, \; N = 430$$

whence
$$\mu = \frac{422{\cdot}7 \times 440^2 \times 500}{2\pi \times 440 \times 430^2} \times 12$$
$$= 961 \text{ inch lbs. per radian per sec.}$$

Work absorbed (from Equation (44))

$$= \pi \times 961 \times 269 \times (0{\cdot}578\theta)^2$$
$$= 272{,}000\theta^2 \text{ inch lbs.}$$

Equating input to work absorbed, we now have the following :—

$$47500\theta = 4{,}152{,}000\theta^2 + 54{,}500{,}000\theta^{2\cdot8}$$

whence
$$54500\theta^{1\cdot8} + 4152\theta = 47{\cdot}5$$

The solution of this equation, obtained graphically, is

$$\theta = 0{\cdot}0088 \text{ radian}$$
$$= 0{\cdot}504^\circ$$

The actual measured amplitude on test was 0·45°.

It is of interest to construct a balance sheet of the damping. This will read as follows :—

Location of Damping.	Energy Absorbed per Vibration (inch lbs.).	Percentage.
Engine . . .	300·7	71·95
Generator Shaft .	96·2	23·0
Generator . .	21·1	5·05
	418·0	100·00

GENERAL NOTES ON CRITICAL SPEEDS

(i) A CRITICAL speed of appreciable magnitude usually, but not invariably, advertises its presence by a rumbling noise. This noise is caused by the various moving parts—crank journals, pins, gudgeons and pistons—hammering from side to side in their respective clearances, and by backlash in gear teeth. Whether an unusual rumbling noise is, or is not, due to a critical speed, can be determined by running the engine overspeed, cutting out the fuel pumps and allowing the engine to "run down." If the engine runs quietly over the greater part of the range, and the noise occurs at clearly defined points, then the trouble is almost certainly due to resonant torsional vibration.

(ii) The noise produced in a critical speed depends on the temperature of the lubricating oil. An engine may run quietly over its whole range for a short time after starting up, but when the lubricating oil has warmed up and become much less viscous, the noise can appear.

(iii) Noise is no criterion of the stress set up in the shafting. An engine with the flywheel located at the camshaft end is invariably less noisy in a critical speed than the same engine with the flywheel at the opposite end to the camshaft drive, yet the stress may be equal in each case. While engines of the former class may run quite safely with torsional stresses of ±2·5 to ±3·0 tons per sq. in., engines of the latter class must have these stresses reduced to ±1·0 ton per sq. in. on account of noise.

(iv) The amount of noise produced in a critical speed is also dependent on the shape of the normal elastic curve. Fig. 35 shows two typical normal elastic curves for diesel generator sets. In Example (*a*) the motion of the crankshaft is great in comparison with the generator end of the system, whilst the reverse is the case in Example (*b*); (*a*) will therefore be noisy on account of the movement of the engine parts in their respective clearances, while (*b*) will be comparatively quiet as the greater part of the movement takes place in the generator where purely torsional movement has no effect on bearing clearances.

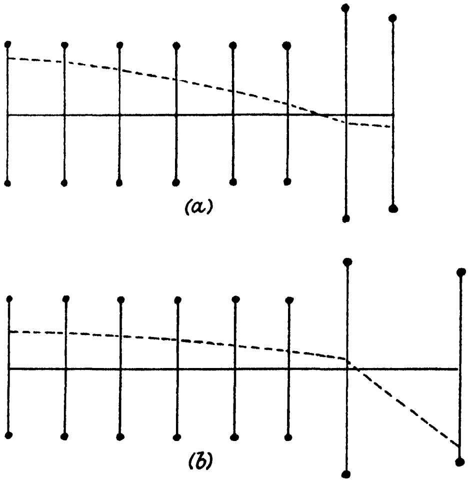

Fig. 35.—Types of One-noded Normal Elastic Curves.

In two particular cases which were recently investigated, a diesel-electric set of type (*a*) was excessively noisy in a sixth order critical speed when the measured stress was ±1·5 tons per sq. in. The second, of type (*b*), had a stress of ±6·3 tons per sq. in. in the generator shaft, and the critical speed was almost inaudible.

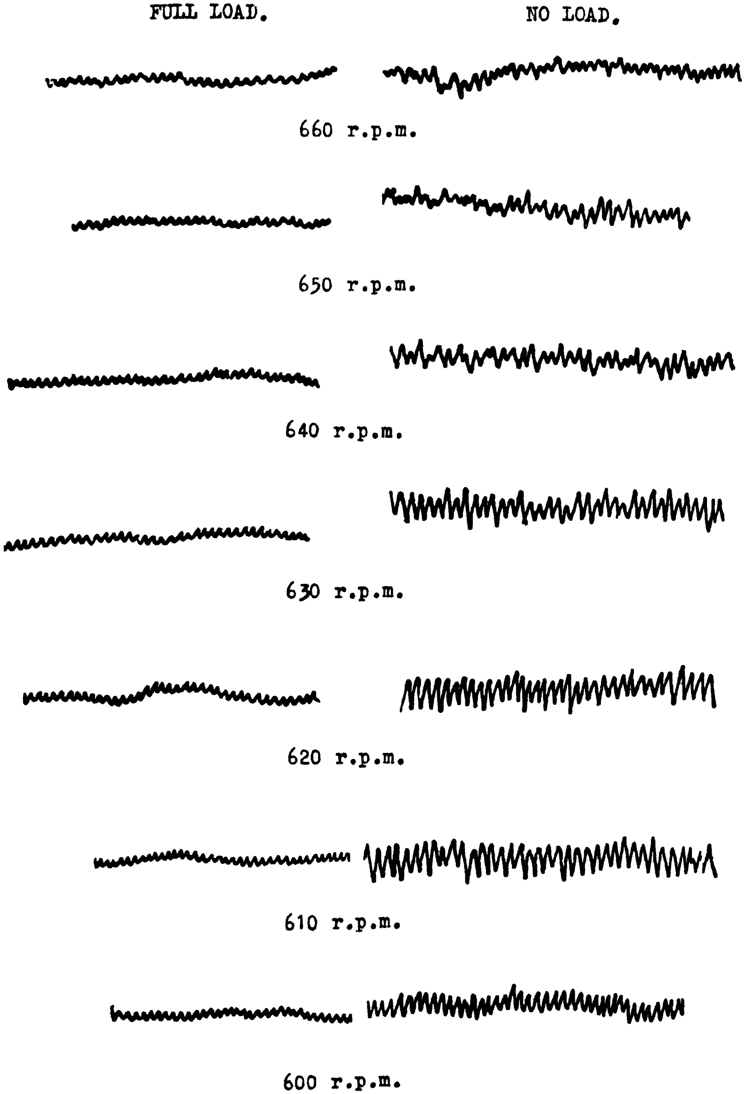

Fig. 36.—Effect of Critical Speed on Speed Variation.

(v) The effect of a critical speed on governing is of great importance. An engine, on full load, running at a critical speed, will run fairly steadily, yet, immediately the load is dropped off, the governor will surge violently. This fact is explained as follows. A critical speed requires energy to maintain it, and this energy is taken from the normal developed output of the engine. In addition, the energy reaches a peak at the critical speed, falling away on either side. The governor is therefore faced with the difficulty of controlling the engine on a load which falls away whether the speed rises or falls. It is incapable of performing this duty, and a surge is the result. The trouble is most noticeable at light load, as the "critical speed energy" is an appreciable proportion of the normal developed energy. At full load it is negligible.

Fig. 36 shows a series of tachograph records which were taken at the same time as the torsiograph test on the engine quoted in the example on page 70. The governor surge at light load, in the sixth order critical at 626 r.p.m., is clearly marked.

VIBRATION MEASURING INSTRUMENTS

THE measurement of torsional vibration is an essential part of the complete investigation of vibration problems in diesel engines. Lack of information concerning the various damping factors renders impossible the prediction of vibration amplitudes from first principles, and it is only by "knowing the answer," in the form of direct measurement, that to-day's knowledge of damping is greater than it was some years ago.

The instrument used is known as a torsiograph or torsional vibrograph and is actually a type of seismograph arranged to operate torsionally. The general principle is shown in Fig. 37. A light pulley *a*, is driven from the engine crankshaft in such a way that it faithfully follows the periodic angular displacements of the latter. The flywheel *b*, mounted on ball bearings and connected to *a* by a light spring or springs, has a sensibly constant angular velocity. By a system of links the relative motion between *a* and *b* is recorded, the record so obtained being a representation of the torsional vibration together with any other uneven motion that may be present.

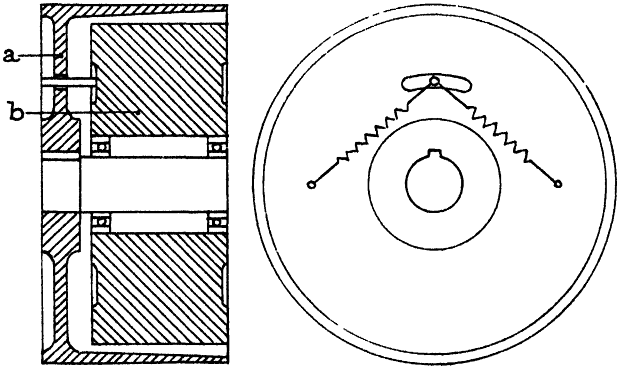

Fig. 37.—Essential Parts of the Torsiograph.

There are a number of types of torsiograph in existence, but it is proposed to describe here only those two types which are commercially in production and which are in general use in this country, viz. the Geiger and the Cambridge.

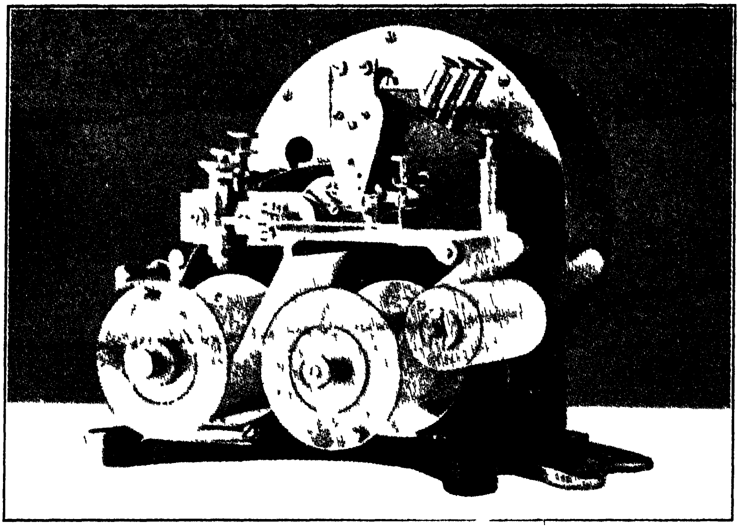

Fig 38.—Geiger Torsiograph (Operating Side).

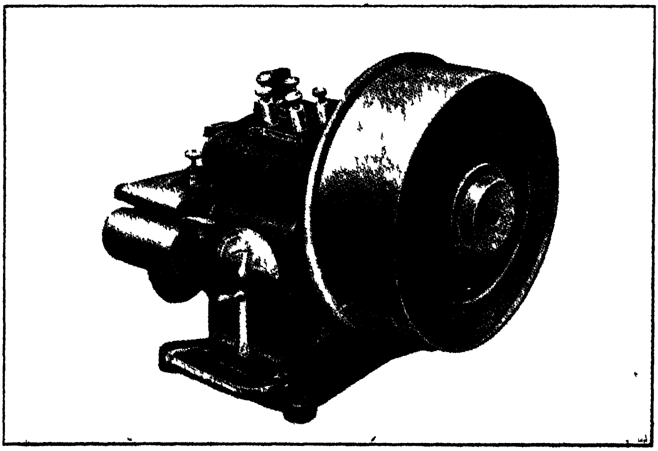

Fig. 39.—Geiger Torsiograph (Pulley Side).

The Geiger Torsiograph.—This instrument is illustrated in Figs. 38 and 39, while Fig. 40 is a sectional view of a part of the Torsiograph showing the principle of operation. It consists of a very light Aluminium Belt Pulley, inside which, and concentric thereto, runs a heavy Flywheel. Both Pulley and Flywheel run independently on Precision Ball Bearings but are connected to each other by a carefully calibrated Volute Spring.

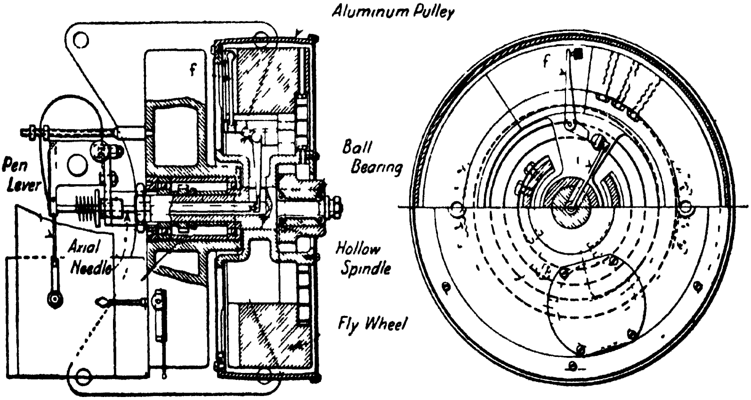

Fig. 40.—Geiger Torsiograph (Section)

When the Torsiograph is in use, the pulley is belt driven from the shaft, the characteristics of which have to be revealed. This pulley thus takes up all the torsional vibrations and irregularities of the shaft, while the flywheel tends to run at a uniform speed. It will thus be seen that there are angular movements relative to pulley and flywheel and these are transmitted by levers to the tracing pen which marks the diagram on a running strip of paper.

The tracing lever is provided with different alternative fulcrums so that the diagram may be obtained in different amplifications according to speed and frequency.

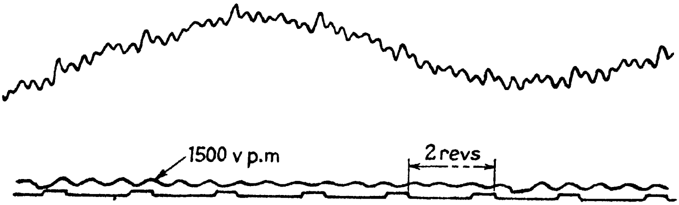

Fig. 41.—Geiger Torsiogram.

The paper band is moved either by gearing from the pulley or by clockwork.

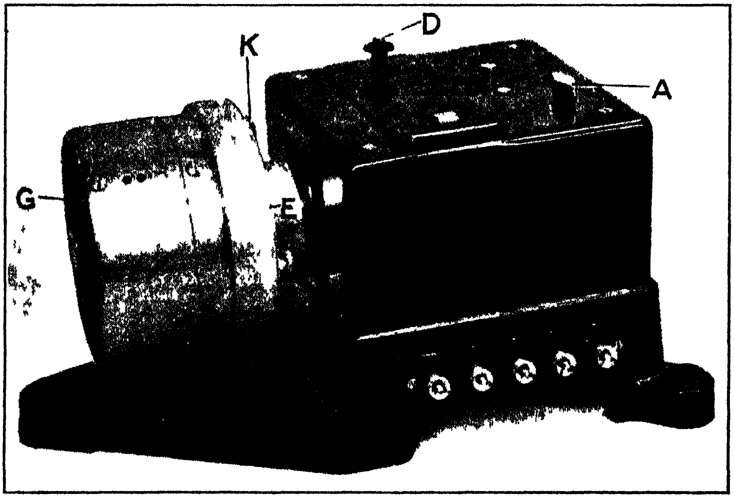

Fig. 42.—Cambridge Torsiograph.

For correctly timing the torsiograms, there is a spring, calibrated to make a certain number of vibrations (usually 1500 per minute) which are marked on the paper through a second pen. A third pen, operated by a contact breaker, either from the injector or the exhaust valve, indicates the revolutions per minute and the phase of the vibrations.

A typical Geiger torsiogram, on which the tracings from the three pens are recorded, is shown in Fig. 41.

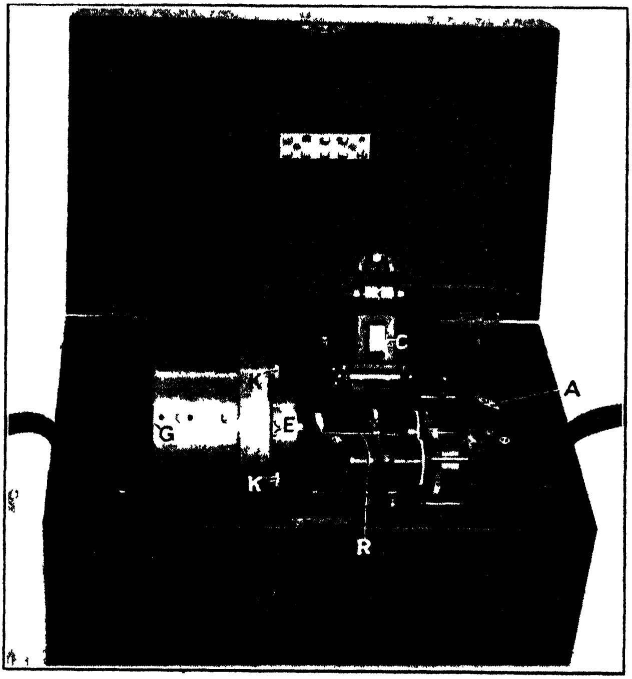

Fig 43 —Cambridge Torsiograph (with Cover removed)

The Cambridge Torsional Vibrograph.—This instrument is shown in Figs. 42 and 43. The general working principle comprising the light aluminium pulley and the floating flywheel, is the same as that of the Geiger, but the method of recording is entirely different.

The instrument employs, as a recording member, a spherical ended stylus tracing on a transparent celluloid strip. The principle is based upon the fact that the elastic limit of such material as celluloid is low, and therefore plastic deformation occurs under comparatively slight pressures. The records obtained are so smooth and fine in character that, when optically magnified, readings accurate to about 0·001 millimetre can be made.

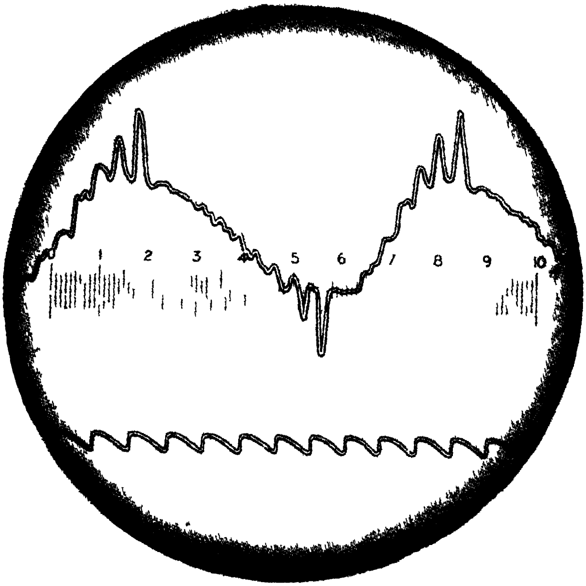

Fig 44 —Cambridge Torsiogram viewed through Microscope.

Instruments recording on this principle need make only small movements, and consequently the controlling forces used in the instrument are relatively great and a high frequency response can be obtained, whilst inertia reactions are small.

Fig. 44 shows the appearance of the record when viewed through a simple microscope. The formation of the stylus point is important, for if a fine geometrical point were used, it would cut or tear a jagged line and would require the application of comparatively large pressures. It is important that no material be cut or torn from the surface.

For correctly timing the torsiograms, a 1/10th second chronometer operates a second stylus, whilst a third stylus is operated by contact breaker from the injector or exhaust valve.

To facilitate the examination and enlargement of the torsiograms, the portable viewer and enlarger shown in Fig. 45 is used. An enlarged view of the record (magnified 10 times) is obtained on a focussing screen measuring 5¼ × 3¼ inches. On this screen are ruled horizontal lines giving the scale of angular deflection in degrees.

The focussing screen can be replaced by a camera dark slide, enabling photographic enlargements to be obtained on either a plate negative or on bromide paper. From the plate negative, a further enlargement can be made if desired, so that records of the original, magnified 100 times, can easily be obtained. This is particularly valuable when a minute examination of the wave form is required.

A typical torsiogram taken from a diesel engine is shown in Fig. 46.

Operating the Torsiograph.—As in numerous other branches of engineering, where entirely fallacious conclusions may be drawn as a result of incorrect use of a measuring instrument, results that are of value can be obtained only by knowing, first, the nature of the vibration to be measured, and second, the most suitable

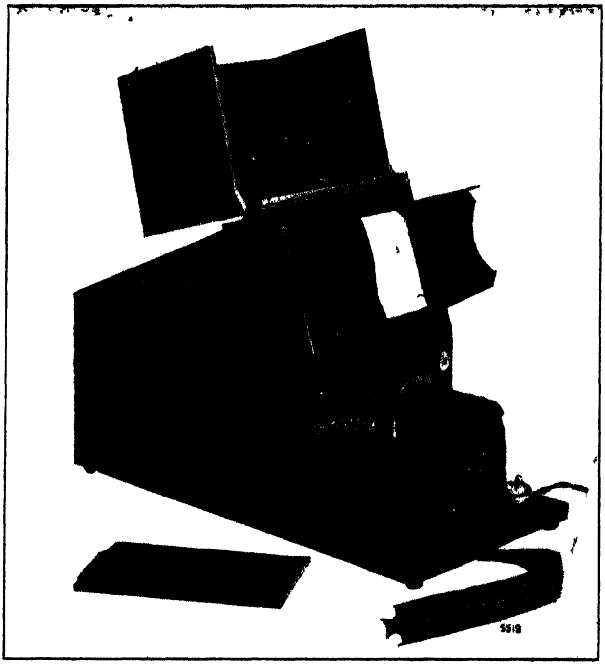

Fig 45.—Cambridge Portable Viewer and Enlarger

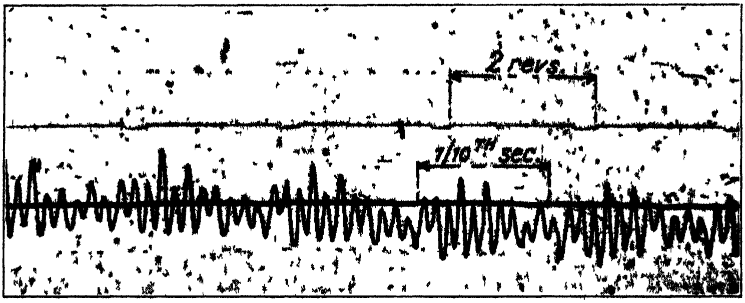

Fig. 46.—Typical Cambridge Torsiogram.

application of the torsiograph. The general procedure can best be set forth as the series of rules given below. Some of these rules are incorporated in the instructions supplied with the instruments, whilst others are based on the author's experience.

(*a*) The torsiograph belt should not be provided with an ordinary belt fastener, but with a butt joint as shown in Fig. 47.

(*b*) The material of the belting should be either the stretchless material supplied by the Geiger Co., or steel tape 0·003 inch thick, the latter being preferable. Ordinary tapes, such as that used for tachometers, should not be used.

(*c*) Pulley centres should be as short as possible, especially in the case of high frequency vibrations. The belt should be run sufficiently tightly to obviate any possibility of slip. In suitable locations, the torsiograph may be direct driven from the end of the shaft.

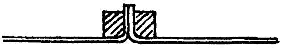

Fig. 47.—Butt Joint Belt Fastener.

(*d*) The ratio of pulley sizes is immaterial. If the pulleys differ in diameter, the scale of the torsiogram should be multiplied by d_1/d_2, where d_1 is the diameter of the torsiograph pulley and d_2 is the diameter of the driving pulley.

(*e*) The vibrations should, in general, be measured at a point of maximum amplitude. This point is obtained from the normal elastic curve.

(*f*) Where the one and two noded natural frequencies are comparatively close together, a diagram taken in compliance with rule (*e*) will give a curve which is a combination of the two modes of vibration and which therefore requires a harmonic analysis to obtain the separate records. Under these conditions it is very helpful to take additional torsiograms from one of the nodes of the two node mode of vibration (see Fig. 48). This clears the torsiogram of all but the one node vibrations.

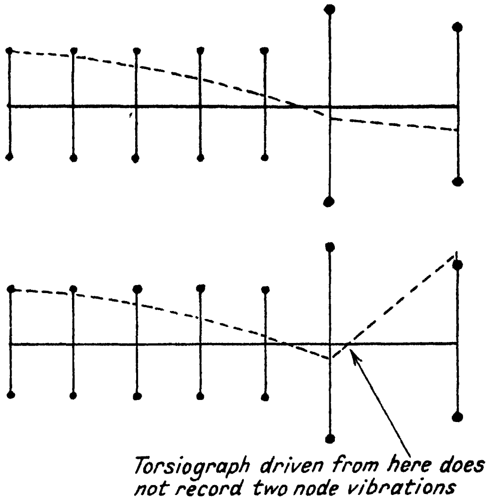
Torsiograph driven from here does not record two node vibrations

Fig. 48.—Method of Obtaining Clear Record of One Noded Vibrations.

It occasionally happens that, when using a torsiograph, the stylus vibrates violently between the stops, as shown in Fig. 49. The usual cause of this phenomenon is the use of too high a magnification, the vibration transmitted from the engine setting up a synchronous vibration of the stylus between the stops.

In cases where a reduction of the magnification does not cure the trouble, an examination of the engine will usually reveal the cause. The fault may be found in a faulty injector or a sticky or bouncing valve, either of which causes, momentarily and periodically reduces the power of one cylinder and so causes a sudden speed variation.

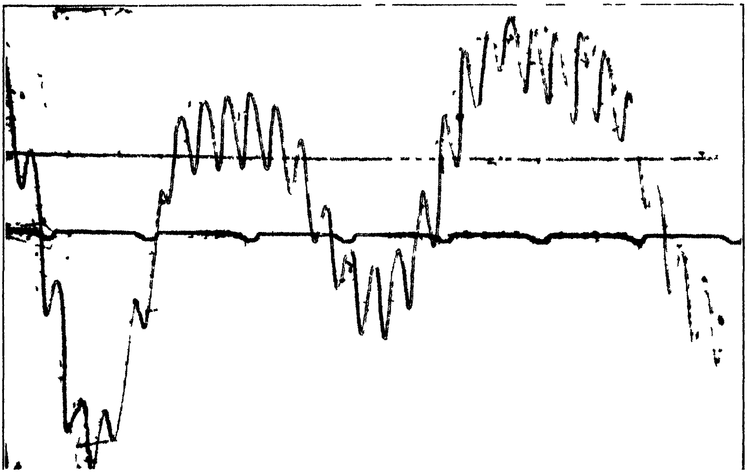

Fig. 49.—Torsiogram showing Violent Instrument Vibration.

THE INTERPRETATION OF THE TORSIOGRAM

HAVING obtained a reliable reading from the instrument it is now necessary that the torsiogram shall be correctly interpreted. In this connection it must be remembered that only in critical speeds which are severe compared with others in their immediate neighbourhood, is the exclusive presence of a particular order of critical indicated by the recording of a pure sine wave. In the majority of cases, the diagram of the critical speed under investigation has, superimposed on it, the out of phase components of neighbouring critical speeds. This phenomenon produces a distortion in the diagram and tends to complicate matters somewhat when engine damping factors are being investigated.

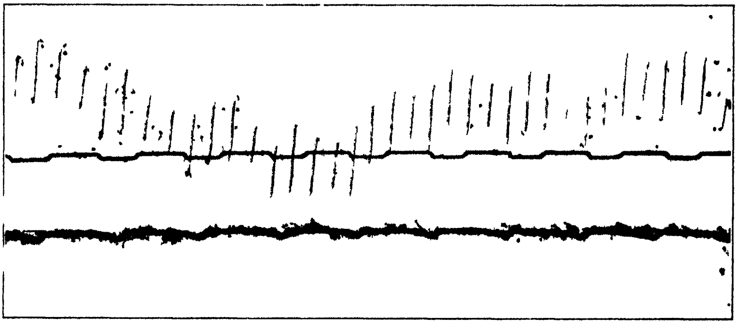

Fig 50 - Torsiogram from Four Cylinder Engine

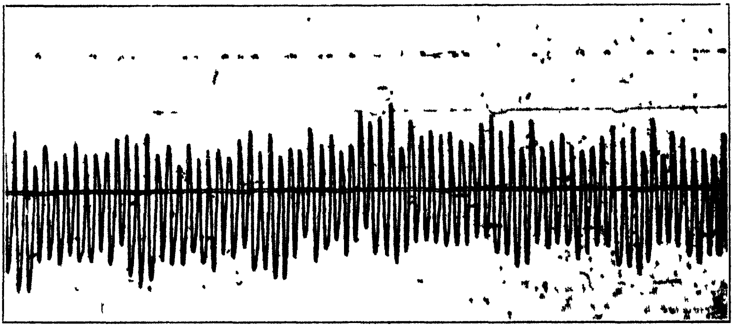

Fig. 51 —Torsiogram from Six Cylinder Engine

Figs. 50 and 51 are two typical torsiograms. The former shows a pure 4th order vibration recorded from a four cylinder engine. The latter, which is recorded from a six cylinder engine, shows a 6th order vibration in which the out of phase components of the 5½th and 6½th order minors produce the distortion mentioned above.

There are two methods of dealing with this latter type of torsiogram when an accurate value for the engine damping factor is being sought. The first consists of analysing the diagram into its constituent harmonics, thereby obtaining a series of pure sine waves. The second consists of the reverse process. By this method a series of torsiograms is taken over a given speed range, and a graph of peak amplitudes is plotted against the speed. Then, assuming a damping factor, the theoretical amplitude curve is built up. If this does not agree with the measured curve, a further damping factor is assumed until agreement is reached.

The latter method appears at first sight to be the more laborious. A number of cases investigated by the author, however, show that, if the Shannon method of calculation is used, and due allowance is made for external damping (if any), exceptionally close agreement between measured and calculated amplitudes is obtained. This being so, the harmonic components of any particular torsiogram can, if required, be read off from the separate components of the theoretical curve and the somewhat lengthy calculation called for by harmonic analysis is thereby avoided.

Figs. 52 and 53 show two typical amplitude-speed curves, the method of constructing which is as follows :—

Using the Shannon method of calculation, the amplitude at each individual critical speed is calculated.

Knowing the dynamic magnifier, Fig. 3 is consulted and the amplitude produced by each individual critical over the entire speed range is then plotted. These are

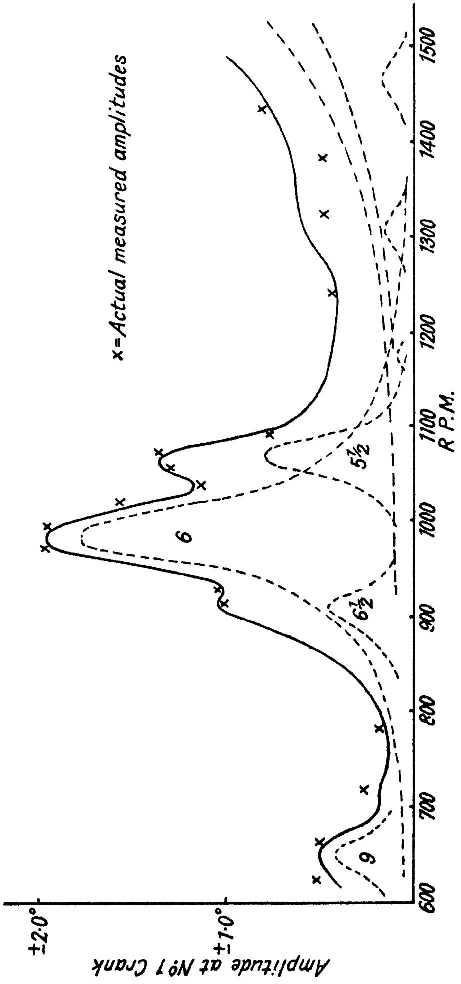

Fig. 52.—Amplitude-Speed Curve, Six Cylinder Diesel Engine.

shown dotted in Figs. 52 and 53. The resultant is then found as the algebraic sum of the ordinates.

In Figs. 52 and 53 the actual measured amplitudes

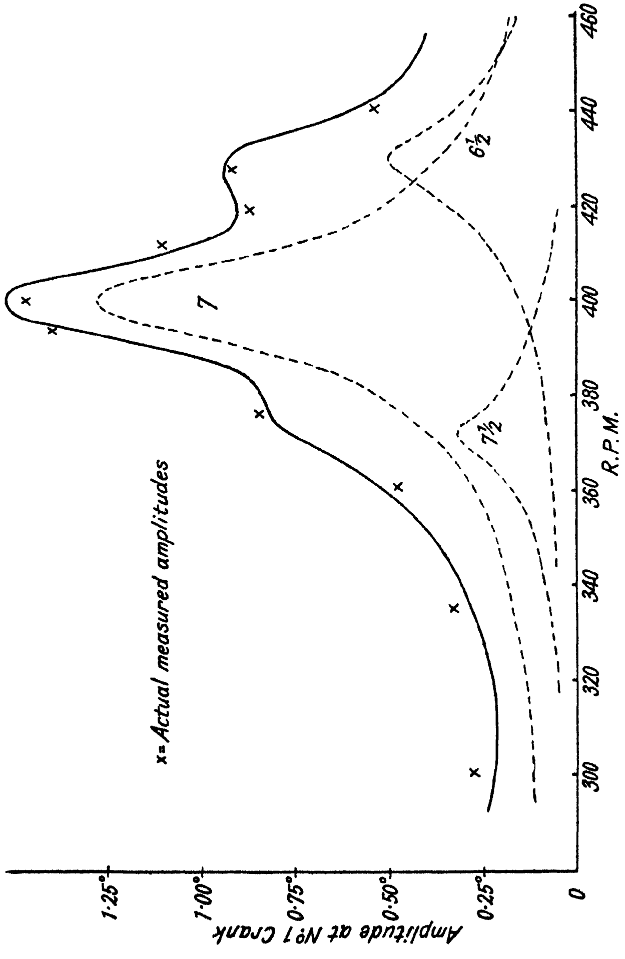

Fig. 53.—Amplitude-Speed Curve, Seven Cylinder Diesel Engine.

have also been plotted and it will be seen that very good agreement has been obtained.

To obtain the stress-speed curve is now a relatively simple matter. From the table for the calculation of the natural frequency of the system, the stress *S* per degree twist at No. 1 crank is known (see Table I). The ordinates of the amplitude-speed curve are thus multiplied by *S* and the required curve is obtained.

It should be appreciated that in what has been said above in connection with the interpretation of the torsiogram, the latter has been assumed to have been taken from the end of the engine opposite to the flywheel. If the torsiogram is taken from elsewhere on the system, it is necessary, before applying the above analysis, to correct it to the amplitude at No. 1 crank by multiplying by the ratio of the ordinates of the normal elastic curve at the two points.

THE ELIMINATION OF TORSIONAL VIBRATION

The estimation of probable critical speeds is to-day one of the most important aspects of crankshaft design. Whilst it may appear a comparatively easy matter first to design the crankshaft and then build the remainder of the engine round it, circumstances often force the hand of the designer and he finds that the crankshaft to fit the job has a dangerous major critical on the running speed.

The problem then arises how to modify the crankshaft to avoid this critical speed without upsetting the remainder of the design. A number of solutions are available, the most important of which are :—

(*a*) Alter the crankshaft stiffness.

(*b*) Alter the rotating and reciprocating masses.

(*c*) Fit a torsional vibration damper.

In the case of (*a*) and (*b*) stiffness should, wherever possible, be increased and mass reduced, so that the critical speed is raised. If the reverse process is adopted, the lowered critical speed has to be run through each time the engine is started up and shut down.

Modifications to crankshaft stiffness usually take the form of increased journal diameters. An increase in the crankpin diameter, whilst effective, has two disadvantages. It increases the weight of the large end, thus partially offsetting the increase in stiffness, and renders more difficult the removal of the connecting rod through the cylinder liner. Alterations to the crankwebs are of comparatively little value as increased weight again goes with increased stiffness.

It should be remembered that alterations to stiffness are most effective near the node, the value decreasing as the distance from the node increases.

Modifications to rotating and reciprocating weights offer scope only in certain circumstances. Boring the crankpins and chamfering the crankwebs are solutions which are almost ineffective when there are no balance weights, but which are most effective when balance weights are fitted. The reason is that in the former case the decrease in weight is to all intents and purposes offset by the reductions in stiffness, whilst in the latter case, to preserve the same degree of balance, an appreciable amount of material can be removed from the outside periphery of the balance weights.

Increasing the weight of the flywheel has little effect if the moment of inertia is sixty or more times that of each cylinder line. In the case of marine engines with small flywheels, alterations to the flywheel mass are very effective.

It is not usually desirable to make alterations to the reciprocating masses. Apart from the fact that only one half of their weight can be assumed to act at crankpin radius, considerations such as engine efficiency and balance dictate that weight shall be kept at a minimum and remedies for critical speeds sought elsewhere.

Alterations to mass are most effective at points remote from the node. Since the node itself is virtually an infinite flywheel, the addition or subtraction of mass at this point is totally ineffective.

The Torsional Vibration Damper.—Whilst it is axiomatic that alterations to avoid critical speeds should be made in the drawing office and not in the shops, the designer is often caught unawares when a stock engine is sold to run at some speed other than standard, or when a normally constant speed engine has to run at varying speeds. In either case a serious major critical may be encountered, and as alterations to mass and stiffness are difficult and sometimes impossible, it becomes necessary to fit a torsional vibration damper.

The damper, which was invented by Dr. Lanchester, consists essentially of a mass driven through friction discs as shown in Fig. 54. The hub *a* carries the driving discs *b*, which are free to slide in longitudinal grooves in the hub. The driven discs *c*, which are free to slide in longitudinal grooves in the rim *d*, are pressed against discs *b*, by means of the springs *e*. The action of the damper is that, should the shaft carrying the hub be subjected to violent vibration, slipping occurs between the discs *b* and *c*, and the vibration is thereby damped down by viscous drag. The damper is, of course, always placed on the shaft at the point of normal maximum amplitude.

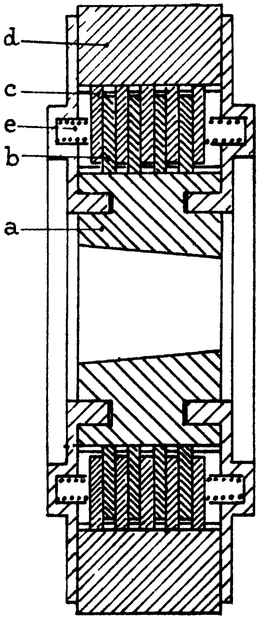

Fig. 54.—Lanchester Damper.

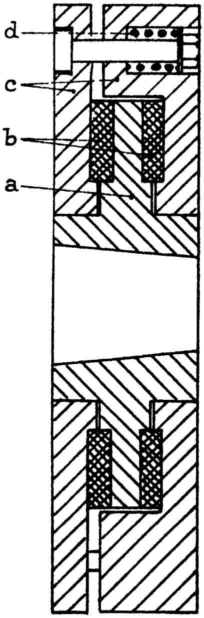

Fig. 55.—Solid Friction Damper.

The design of a damper from first principles presents many difficulties that have yet to be overcome. Major Carter has done much research work on the subject and has published his results,* and whilst these are admittedly incomplete from the point of view of a multi-cylinder diesel engine (the work relates mainly to single crank radial engines) certain valuable conclusions have been reached.

---

* B. C. Carter, "Torsional Vibration in Engines," *Aeronautical Research Committee Report No.* 1053.

When the friction between the plates is zero, the damper has no effect and the position and magnitude of the critical speeds remain unchanged. As the frictional resistance increases, the critical speed becomes lower and the possible amplitude of the vibrations decreases. This effect persists down to a certain point, after which a further increase in frictional resistance reduces still more the critical speed, but increases the possible amplitude of the vibrations, so that when the plates are solidly locked together, the damper becomes an added flywheel and the possible vibration amplitude returns to its original value.

In order that the damper may be effective, its moment of inertia should be at least one half of that of the total crank masses. The effectiveness may be increased by using a suitable connecting shaft stiffness, or by increasing the moment of inertia.

In theory, the Lanchester damper is of the " viscous friction " type, that is, the friction is independent of spring pressure but is proportional to the area of the friction discs. For this reason many of the earlier dampers were provided with a large number of discs and no means of adjusting the springs.

Accumulated experience has, however, shown that the effectiveness of this type of damper can be varied by means of spring pressure. Thus the friction is not " viscous " but approaches the state known as " solid," *i.e.* independent of the slipping velocity of the rubbing members but proportional to the pressure between them.

This experience has led designers to search for a simpler and cheaper design and the outcome is that for medium and high speed engine work, the type shown in Fig. 55 is almost universally adopted.

In this damper, the friction medium consists of two Ferobestos discs *b* which are clamped between the hub *a* and the two discs *c*. The discs *c* are free to rotate on the hub, and the latter is firmly keyed to the shaft. The pressure of the discs *c* on the friction medium is controlled by the adjustable springs *d*.

The action of this damper differs from that of the pure viscous friction type in that whereas the latter is always slipping so long as there is any vibration, the solid friction type acts as an added flywheel until the vibration torque exceeds the static torque setting, at which point slipping begins. It is thus possible, with a constant speed engine having a major critical at the running speed, to fit a solid friction damper set in such a way that it acts as an added flywheel at normal speed, and slips only when starting up and shutting down.

Expressions which greatly facilitate the design of the solid friction damper have been evolved by Den Hartog.* They are as follows :—

Energy dissipated per cycle

$$\delta w = \frac{1}{3\pi} I_d \omega^2 \theta^2 \text{ ft. lbs.} \qquad \ldots \qquad (58)$$

and the optimum friction torque setting at which this dissipation is a maximum

$$T_{opt.} = \frac{\sqrt{2}}{12\pi} I_d \omega^2 \theta \text{ ft. lbs.} \qquad \ldots \qquad (59)$$

---

* "*Mechanical Vibrations*," by Den Hartog.

where I_d = the moment of inertia of the slipping portion of the damper in lbs. ft.^{2} sec.^{2}

ω = phase velocity in radians per sec.

θ = vibration amplitude at the point of application of the damper in radians.

Assume now that a damper is to be fitted at the free end of a diesel engine crankshaft and that the stiffness of the connecting shaft is comparatively high, *i.e.* the vibration amplitude at the damper is the same as that at No. 1 crank.

The input work per cycle from the harmonic forces, from Equation (46), is

$$\delta w = \pi H_n \theta \Sigma a \text{ ft. lbs.}$$

The energy dissipated per cycle by the damper is

$$\delta w_d = \frac{1}{3\pi} I_d \omega^2 \theta^2 \text{ ft. lbs.} \qquad . \quad . \ (60)$$

If engine damping be neglected, *i.e.* if the damper is designed to do the whole of the vibration control, $\delta w = \delta w_d$ and

$$I_d = \frac{3\pi^2 H_n \Sigma a}{\theta \omega^2} \text{ lbs. ft.}^2 \text{ sec.}^2 \qquad . \quad . \ (61)$$

From the frequency calculation table the stress per degree twist at all parts of the shaft being given by the last column, a value of θ is chosen so that the maximum stress at the node will not exceed a certain predetermined value. By successive approximation Σa, ω^2 and I_d can now be calculated for any value of H_n, *i.e.* for any order of critical speed.

Once I_d and θ have been fixed, $T_{opt.}$ can be calculated direct from Equation (59).

There are two points of particular interest in Equation (61) above. The first is that I_d is inversely proportional to θ, which means that to halve the stress it is necessary to double the inertia of the damper if optimum conditions are to be maintained. The second is that I_d is proportional to $H_n \Sigma a$. This means that a damper can be designed for one critical speed only. For all others, optimum conditions cannot be maintained.

Although this latter point may appear at first sight to present insurmountable difficulties, experience shows that, provided the damper is set to deal with the most severe critical in the running range, no trouble from the other criticals need be anticipated.

The Bibby Detuner.—The Bibby detuner is a device which, instead of damping out the torsional vibrations, detunes the system in such a way that resonance conditions cannot build up. It is, in effect, an ingenious adaptation of the well-known Bibby coupling.

This coupling, which is shown part-sectioned in Fig. 56, consists essentially of driving and driven halves *a* and *b*, each provided with teeth and coupled together by means of a continuous spring *c*, which is wound between the teeth in the manner shown. The teeth are flared at the ends *d* in a predetermined manner, thus allowing more clearance for the spring at these points.

The principle of operation is that, when no torque is being transmitted, the spring takes up its normal position with its members parallel to the driving shaft, as shown in Fig. 56. Under these conditions, the coupling has a torsional stiffness which is dependent on the length of spring between the outer ends of the teeth *e*. When overload is being transmitted, the driving half of the coupling moves forward axially relative to the driven half, and the spring lies along the flared flanks of the teeth for their entire length. In this case the torsional stiffness depends on the length of spring between the inner ends of the teeth. For normal working load, an intermediate spring position and torsional stiffness is used.

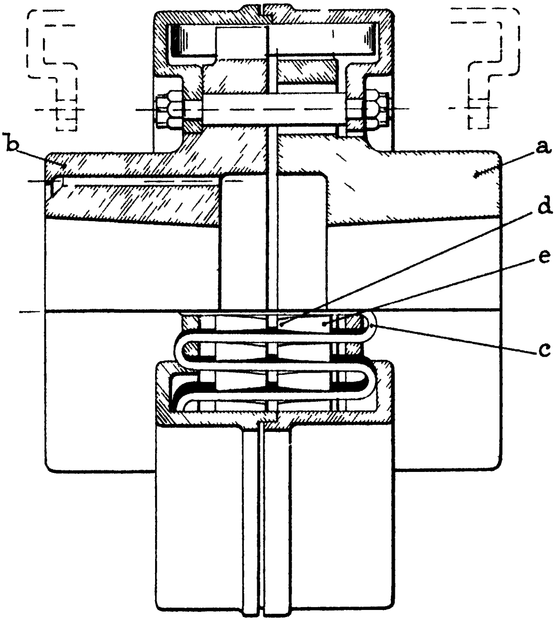

Fig 56 — Bibby Coupling

The coupling thus possesses the unique property of changing its torsional stiffness as the load and consequent angular deflection changes.

The detuner, in one of its forms, consists of a flywheel mounted on the engine crankshaft through a Bibby coupling, in the position normally occupied by a damper.

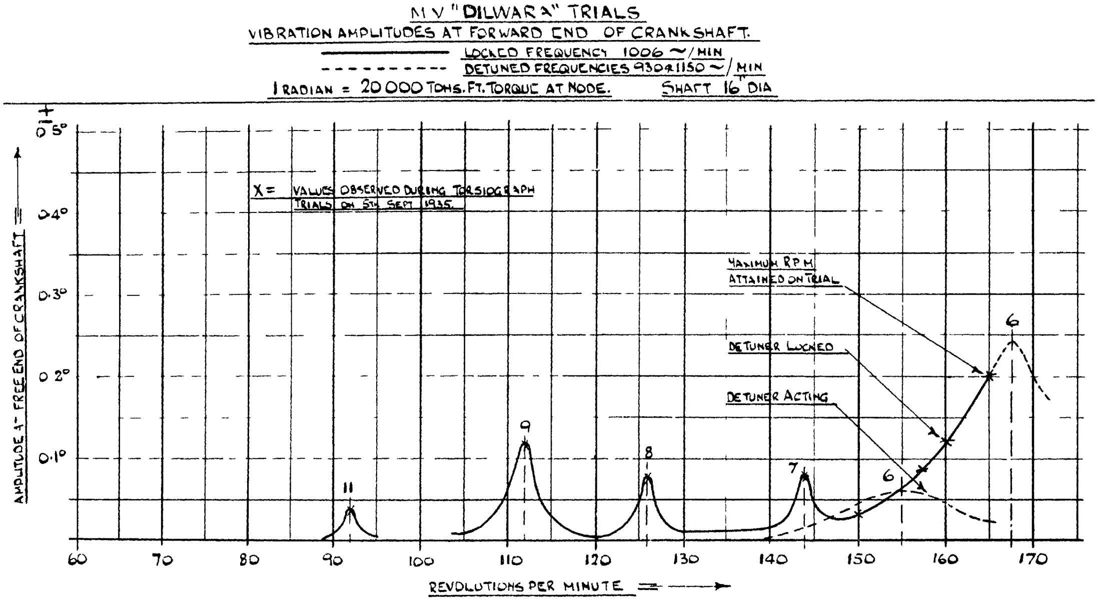

Fig. 57 Amplitude-Speed Curve showing effect of fitting a Bibby Detuner

The above mentioned property of the spring gives a progressive change in the torsional stiffness of the system as the amplitude increases. Torsional resonance is therefore " detuned " as soon as it attempts to build up. The spring characteristic must be designed for the particular system to which the detuner is applied, and it is possible, by careful design, to detune two and sometimes three frequencies.

In Fig. 57 is shown the amplitude-speed curve for the crankshaft of a diesel engine. The effect of fitting a detuner is clearly shown.

ELASTIC CONSTANTS

Material	Torsional Elastic Modulus, G	
	lbs per sq. in.	lbs. per sq ft.
Steel, ...	$11·8 \times 10^6$	$17·0 \times 10^8$
Cast Iron, . . .	$6·3 \times 10^6$	$9·1 \times 10^8$
Wrought Iron, . .	$11·0 \times 10^6$	$15·8 \times 10^8$
Immadium Bronze, . .	$5·0 \times 10^6$	$7·2 \times 10^8$

SAFE WORKING STRESSES IN REVERSED TORSION

Mild Steel, . .	± 2·5	tons	per sq. in.
Nickel Chrome Steel,	± 3·5	,,	,,
Wrought Iron, .	± 2·0	,,	,,
Cast Iron, . .	± 1·0	,,	,,
Immadium Bronze, .	± 1·5	,,	,,

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